5.3. **Determinants.**

Definition. Let $\mathbb{A} \in M(n \times n)$. The symbol $\mathbb{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix, which is created from $\mathbb{A}$ by omitting the i-th row and the j-th column.

Definition. Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$. *Determinant* of the matrix $\mathbb{A}$ is defined by

$$\det \mathbb{A} = \begin{cases} a_{11} & \text{if } n = 1, \\ \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \mathbb{A}_{i1} & \text{if } n > 1. \end{cases}$$

For $\det \mathbb{A}$ we will use also the symbol

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Remark. Let $\mathbb{A} \in M(n \times n)$. If the matrix $\mathbb{A}$ has a zero row or a zero column, then $\det \mathbb{A} = 0$.

Definition. Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$. We say that $\mathbb{A}$ is an *upper triangular matrix* if we have $a_{ij} = 0$ for $i > j$, $i, j \in \{1, \dots, n\}$. We say that $\mathbb{A}$ is a *lower triangular matrix*, if we have $a_{ij} = 0$ for $i < j$, $i, j \in \{1, \dots, n\}$.

Theorem 5.9 (determinant of a triangular matrix)**.** *Let $\mathbb{A} = (a_{ij})_{i,j=1..n}$ be an upper (lower, respectively) triangular matrix. Then we have*

$$\det \mathbb{A} = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}.$$

Theorem 5.10. *Let $j, n \in \mathbf{N}$, $j \leq n$, and matrices $\mathbb{A}, \mathbb{B}, \mathbb{C} \in M(n \times n)$ coincide at each row except j-th row. Let j-th row of $\mathbb{A}$ be equal to the sum of j-th rows of $\mathbb{B}$ and $\mathbb{C}$. Then we have* $\det \mathbb{A} = \det \mathbb{B} + \det \mathbb{C}$.

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1+v_1 & \dots & u_n+v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1 & \dots & u_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ v_1 & \dots & v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Theorem 5.11 (determinant and elementary row transformations)**.** *Let $\mathbb{A}, \mathbb{A}' \in M(n \times n)$.*

(i) Let $\mathbb{A}'$ be created from $\mathbb{A}$ such that we interchanged two rows in $\mathbb{A}$ (i.e., we applied an elementary row transformation of the first kind). Then we have $\det \mathbb{A}' = -\det \mathbb{A}$.

(ii) Let $\mathbb{A}'$ be created from $\mathbb{A}$ such that a row in $\mathbb{A}$ is multiplied by $\lambda \in \mathbf{R}$. Then we have $\det \mathbb{A}' = \lambda \det \mathbb{A}$.

(iii) Let $\mathbb{A}'$ be created from $\mathbb{A}$ such that we added a multiple of a row of $\mathbb{A}$ to another row of $\mathbb{A}$ (i.e., we applied an elementary row transformation of the third kind). Then we have $\det \mathbb{A}' = \det \mathbb{A}$.

Corollary 5.12. *Let $\mathbb{A}, \mathbb{A}' \in M(n \times n)$ and $\mathbb{A}'$ be created from $\mathbb{A}$ applying a transformation. Then* $\det \mathbb{A}' \neq 0$ *if and only if* $\det \mathbb{A} \neq 0$.

More precisely, for any transformation T of n-by-n matrices there is $\alpha \in \mathbf{R} \setminus \{0\}$ such that whenever $\mathbb{A}, \mathbb{A}' \in M(n \times n)$ and $\mathbb{A}'$ is created from $\mathbb{A}$ using T, then $\det \mathbb{A}' = \alpha \cdot \det \mathbb{A}$.

Theorem 5.13 (determinant and invertibility). *Let* $\mathbb{A} \in M(n \times n)$. *Then* $\mathbb{A}$ *is invertible if and only if* $\det \mathbb{A} \neq 0$.

Theorem 5.14 (determinant of product). *Let* $\mathbb{A}, \mathbb{B} \in M(n \times n)$. *Then we have*

$$\det \mathbb{A}\mathbb{B} = \det \mathbb{A} \cdot \det \mathbb{B}.$$

Remark. The analogue of Theorem 5.10 holds also for columns in place of rows. The analogue of Theorem 5.11 holds also for elementary column transformations. The analogue of Corollary 5.12 hold also for column transfromation. (The proof of the analogue of Theorem 5.11(i) is a bit more complicated.)

Theorem 5.15 (determinant of a transpose). *Let* $\mathbb{A} \in M(n \times n)$. *Then we have* $\det \mathbb{A}^T = \det \mathbb{A}$.

Theorem 5.16 (Laplace's formula). *Let* $\mathbb{A} = (a_{ij})_{i,j=1..n}$, $k \in \{1, \ldots, n\}$. *Then*

$$\det \mathbb{A} = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det \mathbb{A}_{ik},$$

$$\det \mathbb{A} = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det \mathbb{A}_{kj}.$$