4. Functions of several variables

4.1. $\mathbf{R}^n$ as a metric and linear space.

Definition. The set $\mathbf{R}^n$, $n \in \mathbf{N}$, is the set of all ordered n-tuples of real numbers, i.e.

$$\mathbf{R}^n = \{[x_1, \dots, x_n] : x_1, \dots, x_n \in \mathbf{R}\}.$$

For $\boldsymbol{x} = [x_1, \dots, x_n] \in \mathbf{R}^n$, $\boldsymbol{y} = [y_1, \dots, y_n] \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$ we set

$$\boldsymbol{x} + \boldsymbol{y} = [x_1 + y_1, \dots, x_n + y_n], \qquad \alpha\boldsymbol{x} = [\alpha x_1, \dots, \alpha x_n].$$

Further, we denote $\boldsymbol{o} = \mathbf{0} = [0, \dots, 0]$ – the *origin*.

Definition. *Euclidean metric on* $\mathbf{R}^n$ is the function $\rho\colon \mathbf{R}^n \times \mathbf{R}^n \to [0, +\infty)$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

The number $\rho(\boldsymbol{x}, \boldsymbol{y})$ is called *distance of the point $\boldsymbol{x}$ from the point $\boldsymbol{y}$*.

Theorem 4.1 (properties of Euclidean metric)**.** *Euclidean metric ρ has the following properties:*

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,
(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*
(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbf{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) \le \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*
(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n, \forall \lambda \in \mathbf{R}\colon \rho(\lambda\boldsymbol{x}, \lambda\boldsymbol{y}) = |\lambda|\, \rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*
(v) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbf{R}^n\colon \rho(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + \boldsymbol{z}) = \rho(\boldsymbol{x}, \boldsymbol{y})$. *(translation invariance)*

Definition. Let $\boldsymbol{x} \in \mathbf{R}^n,\ r \in \mathbf{R}, r > 0$. The set $B(\boldsymbol{x}, r)$ defined by

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbf{R}^n;\ \rho(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

is called *open ball with radius r centered at $\boldsymbol{x}$*.

Definition. Let $M \subset \mathbf{R}^n$. We say that $\boldsymbol{x} \in \mathbf{R}^n$ is an *interior point of* M, if there exists $r > 0$ such that $B(\boldsymbol{x}, r) \subset M$. The set of all interior points of M is called the *interior of* M and is denoted by $\operatorname{Int} M$. The set $M \subset \mathbf{R}^n$ is *open in* $\mathbf{R}^n$, if each point of M is an interior point of M, i.e., if $M = \operatorname{Int} M$.

Theorem 4.2 (properties of open sets)**.**

(i) The empty set and $\mathbf{R}^n$ are open in $\mathbf{R}^n$.
(ii) Let sets $G_\alpha \subset \mathbf{R}^n$, $\alpha \in A \neq \emptyset$, be open in $\mathbf{R}^n$. Then $\bigcup_{\alpha \in A} G_\alpha$ is open in $\mathbf{R}^n$.
(iii) Let sets G_i, $i = 1, \dots, m$, be open in $\mathbf{R}^n$. Then $\bigcap_{i=1}^m G_i$ is open in $\mathbf{R}^n$.

Definition. Let $\boldsymbol{x}^j \in \mathbf{R}^n$ for each $j \in \mathbf{N}$ and $\boldsymbol{x} \in \mathbf{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^\infty$ *converges to* $\boldsymbol{x}$, if $\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0$. The vector $\boldsymbol{x}$ is called *limit of the sequence* $\{\boldsymbol{x}^j\}_{j=1}^\infty$.

Theorem 4.3 (convergence is coordinatewise)**.** *Let $\boldsymbol{x}^j \in \mathbf{R}^n$ for each $j \in \mathbf{N}$ and $\boldsymbol{x} \in \mathbf{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^\infty$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \dots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^\infty$ converges to the real number x_i.*

Definition. Let $M \subset \mathbf{R}^n$ and $\boldsymbol{x} \in \mathbf{R}^n$. We say that $\boldsymbol{x}$ is a *boundary point of* M, if for each $r > 0$ we have $B(\boldsymbol{x}, r) \cap M \neq \emptyset$ and $B(\boldsymbol{x}, r) \cap (\mathbf{R}^n \setminus M) \neq \emptyset$.

Boundary of M is the set of all boundary points of M (notation $\operatorname{bd} M$).

Closure of M is the set $M \cup \operatorname{bd} M$ (notation $\overline{M}$).

A set $M \subset \mathbf{R}^n$ is said to be *closed* if it contains all its boundary points, i.e., if $\operatorname{bd} M \subset M$, i.e., if $\overline{M} = M$.

Theorem 4.4 (characterization of closed sets)**.** *Let $M \subset \mathbf{R}^n$. Then the following assertions are equivalent:*

(1) M is closed.
(2) $\mathbf{R}^n \setminus M$ is open.
(3) Any $\boldsymbol{x} \in \mathbf{R}^n$ which is a limit of a sequence from M belongs to M.

Theorem 4.5 (properties of closed sets)**.**

(i) The empty set and $\mathbf{R}^n$ are closed in $\mathbf{R}^n$.
(ii) Let sets $F_\alpha \subset \mathbf{R}^n$, $\alpha \in A \neq \emptyset$, be closed in $\mathbf{R}^n$. Then $\bigcap_{\alpha \in A} F_\alpha$ is closed in $\mathbf{R}^n$.
(iii) Let sets F_i, $i = 1, \ldots, m$, be closed in $\mathbf{R}^n$. Then $\bigcup_{i=1}^{m} F_i$ is closed in $\mathbf{R}^n$.