FUNCTIONAL ANALYSIS 2

SUMMER SEMESTER 2023/2024

PROBLEMS TO CHAPTER X

Problems to Section X.1 – examples of Banach algebras, invertible elements

Problem 1. Let $A = (\mathbb{C}^n, \|\cdot\|_p)$, where $p \in [1, \infty]$ and $n \geq 2$. Equip A with the coordinatewise multiplication, i.e.,

$$(x_1, x_2 \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = (x_1 y_1, x_2 y_2, \dots, x_n y_n).$$

(1) Show that A is a unital Banach algebra and find its unit.
(2) Show that the unit has norm one if and only if $p = \infty$.
(3) Apply on A the respective renorming and show, that the new norm is just $\|\cdot\|_\infty$.

Problem 2. Let M_n be the algebra of complex $n \times n$-matrices equipped with the matrix multiplication. Recall that any $n \times n$-matrix represents a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and that the matrix multiplication corresponds to composition of linear mappings.

(1) Fix $p \in [1, \infty]$ and equip M_n with the operator norm coming from $L((\mathbb{C}^n, \|\cdot\|_p))$. Show that M_n is then a unital Banach algebra and that the unit has norm one.
(2) Show that for $p_1 \neq p_2$ the two norms defined in (1) are equivalent but different whenever $n \geq 2$.
(3) Show that M_n is commutative if and only if $n = 1$.

Problem 3. Let M_n be the algebra of complex $n \times n$-matrices equipped with the matrix multiplication. Equip M_n with the norm

$$\|(a_{ij})_{i,j=1,\dots,n}\| = \sum_{i,j=1}^{n} |a_{ij}| \,.$$

Show that M_n equipped with this norm is a unital Banach algebra and its unit has norm greater than 1 (whenever $n \geq 2$).

Problem 4. Let $A = (\mathbb{C}^n, \|\cdot\|_\infty)$, where $n \geq 2$.

(1) Define multiplication on A by

$$(x_1, x_2 \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = (x_1 y_1, x_1 y_2, \dots, x_1 y_n).$$

Show that A equipped with this mutliplication is a Banach algebra and that A has many left units but no right unit.

(2) Define multiplication on A by

$$(x_1, x_2 \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = (x_1 y_1, x_2 y_1, \dots, x_2 y_1).$$

Show that A equipped with this mutliplication is a Banach algebra and that A has many right units but no leftt unit.

(3) Represent the algebras from (1) and (2) as subalgebras of the matrix algebra M_n.

`Hint:` *(3) Consider matrices with only one nonzero row or column.*

Problem 5. Let $A = \ell^p(\Gamma)$, where $p \in [1, \infty]$ and Γ is an infinite set. Equip A with the pointwise multiplication.

(1) Show that A is a Banach algebra.
(2) Show that A is unital if and only if $p = \infty$.

Problem 6. Let X be any nontrivial Banach space. Define on X the trivial multiplication, i.e., $x \cdot y = \boldsymbol{o}$ for $x, y \in X$.

(1) Show that X is a Banach algebra with no unit.
(2) Describe the unital algebra X^+.
(3) Represent the algebras X and X^+ as subalgebras of $L(X^+) = L(X \oplus_1 \mathbb{C})$.
(4) Find a subalgebra of the matrix algebra M_n (where $n \geq 2$) isomorphic with such a trivial algebra.

`Hint:` *(4) Use the description from (3) for* $X = \mathbb{C}^{n-1}$.

Problem 7. Let $A_1, \dots, A_n$ be Banach algebras and let $p \in [1, \infty]$. Consider the vector space $A = A_1 \times A_2 \times \cdots \times A_n$, where the norm and multiplication are defined by

$$\|(a_1, \dots, a_n)\| = \|(\|a_1\|, \dots, \|a_n\|)\|_p ,$$
$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n).$$

(1) Show that A is a Banach algebra.
(2) Show that A is unital if and only if $A_1, \dots, A_n$ are unital. (Find the respective unit.)
(3) Show that A is commutative if and only if $A_1, \dots, A_n$ are commutative.

Problem 8. Let T be a Hausdorff locally compact space. Consider the Banach space $\mathcal{C}_0(T)$ (with the max-norm) and equip it with the pointwise multiplication.

(1) Show that $\mathcal{C}_0(T)$ is a commutative Banach algebra.
(2) Show that the algebra $\mathcal{C}_0(T)$ is unital if and only if T is compact.
(3) Assume that T is not compact. Let $B = \operatorname{span}(\mathcal{C}_0(T) \cup \{1\})$ as a subalgebra of $\ell^\infty(T)$. Show that B is (algebraically) isomorphic to $(\mathcal{C}_0(T))^+$ but not isometric.

Problem 9. Let K be a compact Hausdorff space and let A be a Banach algebra. Let $\mathcal{C}(K, A)$ be the vector space of all the continuous mappings $f : K \to A$. Equip $\mathcal{C}(K, A)$ with the norm and with the multiplication given by

$$\|f\| = \sup\{\|f(t)\| \, ; t \in K\}, \qquad f \in \mathcal{C}(K, A),$$
$$(f \cdot g)(t) = f(t) \cdot g(t), \quad t \in K, \quad f, g \in \mathcal{C}(K, A).$$

(1) Show that $\mathcal{C}(K, A)$ is a Banach algebra.
(2) Show that $\mathcal{C}(K, A)$ is unital if and only if A is unital and find the unit.
(3) Show that $\mathcal{C}(K, A)$ is commutative if and only if A is commutative.

Problem 10. Let X be a Banach space. Consider the space $L(X)$ of all continuous linear operators on X equipped with the operator norm. Define the multiplication on $L(X)$ as composing the operators.

(1) Show that $L(X)$ is a unital Banach algebra and find its unit.
(2) Show that $L(X)$ is commutative if and only if $\dim X = 1$.

`Hint:` *(2) If* $\dim X \geq 2$*, choose* $x_1, x_2 \in X$ *linearly independent.Show that there exist* $x_1^*, x_2^* \in X^*$ *such that* $x_1^*(x_1) = x_2^*(x_2) = 1$ *and* $x_1^*(x_2) = x_2^*(x_1) = 0$*. Consider the operators of the form* $x \mapsto x_i^*(x) x_j$ *and their linear combinations.*

Problem 11. Let X be a Banach space and let $A = K(X)$ is the space of compact linear operators on X, considered as a subspace of $L(X)$.

(1) Show that A is a closed subalgebra of $L(X)$, and so it is a Banach algebra.
(2) Show that A is unital if and only if $\dim X < \infty$.
(3) Show that A is commutative if and only if $\dim X = 1$.
(4) Assume that $\dim X = \infty$. Let $B = \operatorname{span}(A \cup \{I\})$ as a subalgebra of $L(X)$. Show that B is (algebraically) isomorphic to A^+, but not isometric.

`Hint:` *(3) Show that the operators used in Problem 10(2) are compact.*

Problem 12. Let $(G, +)$ be a commutative group. Equip the Banach space $\ell^1(G)$ with the multiplication $*$ defined by

$$(f * g)(x) = \sum_{y \in G} f(y)g(x - y), \qquad f, g \in \ell^1(G).$$

Show that $\ell^1(G)$ is then a unital commutative Banach algebra and find its unit.

Problem 13. Let $(G, \cdot)$ be a non-commutative group. Equip the Banach space $\ell^1(G)$ with the multiplication $*$ defined by

$$(f * g)(x) = \sum_{y \in G} f(y)g(y^{-1}x), \qquad f, g \in \ell^1(G).$$

Show that $\ell^1(G)$ is then a unital non-commutative Banach algebra and find its unit.

Problem 14. Let $A = L^1(\mathbb{R}^n)$ where $n \in \mathbb{N}$ (with the standard norm). Define the mutliplication $*$ on A as the convolution, i.e.,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x - y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n, f, g \in A.$$

(1) Show that A is a commutative Banach algebra.
(2) Show that A has no unit.

`Hint:` *(1) Use properties of the convolution. (2) Assume that g is a unit. Then for each $r > 0$ we have $g * \chi_{B(0,r)} = \chi_{B(0,r)}$. Deduce that $\chi_{B(0,r)}(x) = \int_{B(x,r)} g$ for almost all x, in particular $\int_{B(x,r)} g = 1$ almost everywhere $B(0, r)$. For sufficiently small r derive a contradiction with $g \in L^1(\mathbb{R}^n)$.*

Problem 15. Let $A = L^1([0, 1])$ (with the standard norm). Define the mutliplication $*$ on A as the convolution, i.e.,

$$(f * g)(x) = \int_0^1 f(y)g(x - y \mod 1)\,\mathrm{d}y, \quad x \in [0, 1], f, g \in A.$$

(1) Show that A is a commutative Banach algebra.
(2) Show that A has no unit.

`Hint:` *(1) Use an analogous approach as in proving properties of the convolution. (2) Proceed similarly as in Problem 14(2).*

Problem 16. Let $(G, +)$ be a commutative compact topological group. (I.e., $(G, +)$ is a commutative group equipped with a Hausdorff topology in which the operations $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous, which is moreover compact in this topology.) Let $\mathcal{M}(G)$ be the space of all the complex Radon measures on G, equipped with the total variation norm and with the multiplication $*$ defined by

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in G \times G; x + y \in A\}),$$

where $\mu \times \nu$ denotes the respective product measure. Show that $\mathcal{M}(G)$ is then a unital commutative Banach algebra and find its unit.

Problem 17. Let $(G, \cdot)$ be a non-commutative compact topological group. (I.e., $(G, \cdot)$ is a non-commutative group equipped with a Hausdorff topology in which the operations $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are continuous, which is moreover compact in this topology.) Let $\mathcal{M}(G)$ be the space of all the complex Radon measures on G, equipped with the total variation norm and with the multiplication $*$ defined by

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in G \times G; x \cdot y \in A\}),$$

where $\mu \times \nu$ denotes the respective product measure. Show that $\mathcal{M}(G)$ is then a unital non-commutative Banach algebra and find its unit.

Problem 18. Show that in the matrix algebra M_n an element has a right inverse if and only if it has a left inverse.

Problem 19. Let $A = L(\ell^2)$. Define two operators $S, T \in A$ by

$$S(x_1, x_2, \dots) = (x_2, x_3, \dots) \qquad \text{and} \qquad T(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

(1) Show that S and T are not invertible.
(2) Show that S has a right inverse and describe all its right inverses.
(3) Show that T has a left inverse and describe all its left inverses.

Problem 20. Let $G = (\mathbb{Z}_n, +)$ where $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ equipped with the addition modulo n. Let $A = \ell^1(G)$ be the Banach algebra described in Problem 12.

(1) Represent A as a subalgebra of the matrix algebra M_n (with an appropriate norm).
(2) For $n = 2$ and $n = 3$ explicitly characterize invertible elements in A.

`Hint:` *(1) Embed A into $L(A)$.*

Problem 21. Let A be a Banach algebra. Define on A a new multiplication $\odot$ by

$$x \odot y = y \cdot x, \quad x, y \in A.$$

(1) Show that $A^{op} = (A, \odot)$ is a Banach algebra.
(2) Show that A^{op} need not be (algebraically) isomorphic to A.
(3) Let X be a reflexive Banach space. Show that $L(X)^{op}$ is isometrically isomorphic to $L(X^*)$.
(4) Let H be a Hilbert space. Show that $L(H)^{op}$ is isometrically isomorphic to $L(H)$.

`Hint:` *(2) Use Problem 4.*

Problems to Section X.2 – spectrum and its properties

Problem 22. Let $A = \mathcal{C}(K)$ for a compact Hausdorff space K and let $f \in A$.

(1) Show that $\sigma(f) = f(K)$.
(2) Compute the resolvent function of f.

Problem 23. Let $A = \mathcal{C}_0(T)$ for a noncompact locally compact space T.

(1) Show that $\sigma(f) = f(T) \cup \{0\}$ for each $f \in A$.
(2) Suppose that T is not σ-compact. Show that $\sigma(f) = f(T)$ for each $f \in A$.
(3) In case $T = \mathbb{R}$ find an example of $f \in A$ with $f(T) \subsetneqq \sigma(f)$.

`Hint:` *(1) Use the definition $\sigma_A(f) = \sigma_{A^+}(f)$ and a description of A^+ (for example that from Problem 8(3)). (2) $\{t \in T; f(t) \neq 0\}$ is σ-compact.*

Problem 24. Let $A = \ell^1(\mathbb{Z}_n)$ (see Problem 20) and $x \in A$.

(1) Characterize $\sigma(x)$ as the set of eigenvalues of certain matrix.
(2) For $n = 2, 3$ compute $\sigma(x)$ and the resolvent function explicitly.

`Hint:` *Use the solution of Problem 20.*

Problem 25. Let $\mathbb{T} = \{z \in \mathcal{C}; |z| = 1\}$, $A = \mathcal{C}(\mathbb{T})$ and $f(z) = z$ for $z \in \mathbb{T}$. Let B be the unital closed subalgebra of A generated by f, i.e.,

$$B = \overline{\operatorname{span}}\{1, f, f^2, f^3, \dots\}.$$

Compute and compare $\sigma_A(f)$ and $\sigma_B(f)$.

Problem 26. Let A be a unital Banach algebra and let $x \in A$ be such that $x^n = \boldsymbol{o}$ for some $n \in \mathbb{N}$. Determine $\sigma(x)$ and compute the resolvent function.

Problem 27. Let A be a unital Banach algebra and let $x \in A$ be such that $x^2 = x$. Determine $\sigma(x)$ and compute the resolvent function.

`Hint:` *Distinguish three cases: $x = \boldsymbol{o}$, $x = e$ and $x \notin \{\boldsymbol{o}, e\}$. The inverse of $\lambda e - x$ find in the form $\alpha e + \beta x$ for suitable $\alpha, \beta \in \mathbb{C}$.*

Problem 28. Let A be a unital Banach algebra and let $x \in A$ be such that $x^3 = x$. Determine $\sigma(x)$ and compute the resolvent function.

`Hint:` *There are several cases to be distinguished: The case $x^2 = x$ is covered by Problem 27. The case $x^2 = -x$ can be solved similarly as Problem 27. The next case to be solved is $x^2 = e$. Finally, if $x^2 \notin \{e, x, -x\}$, then show that e, x, x^2 are linearly independent and find the inverse of $\lambda e - x$ as a linear combination of e, x, x^2.*

Problem 29. Let $A = \ell^1(\mathbb{Z})$ (cf. Problem 12) and $n \in \mathbb{Z}$, $n \neq 0$. Show that $\sigma(\boldsymbol{e}_n) = \mathbb{T}$ (where $\boldsymbol{e}_n$ is the respective canonical vector) and that

$$R(\lambda, \boldsymbol{e}_n) = \begin{cases} \sum_{k=0}^{\infty} \frac{\boldsymbol{e}_{kn}}{\lambda^{k+1}}, & |\lambda| > 1, \\ \sum_{k=1}^{\infty} -\lambda^k \boldsymbol{e}_{-kn}, & |\lambda| < 1. \end{cases}$$

`Hint:` *This can be proved directly by solving the equation $(\lambda \boldsymbol{e}_0 - \boldsymbol{e}_n) * f = \boldsymbol{e}_0$. One can also use the formula from Proposition IV.8(v) and its modifications.*

Problem 30. Let A be a unital Banach algebra and let f be an entire function. Let

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \qquad \lambda \in \mathbb{C},$$

be its Taylor expansion. Show that for each $x \in A$ we have

$$\tilde{f}(x) = \sum_{n=0}^{\infty} a_n x^n.$$

`Hint:` *Since f is an entire function, in the formula defining the holomorphic calkulus we may integrate along a circle centered at zero with a sufficiently large radius.*

Problem 31. Let $A = M_n$ and $D \in A$ be a diagonal matrix, with values $d_1, \dots, d_n$ on the diagonal.

(1) Show that $\sigma(D) = \{d_1, \dots, d_n\}$ and compute the resolvent function.
(2) Let f be a function holomorphic on a neighborhood of $\sigma(D)$. Show that $\tilde{f}(D)$ is the diagonal matrix with values $f(d_1), \dots, f(d_n)$ on the diagonal.
(3) Deduce that in this case the value of $\tilde{f}(D)$ depends only on $f|_{\sigma(D)}$.

`Hint:` *(2) Consider a cycle consisting of sufficiently small circles with centers $d_1, \dots, d_n$ and use the Cauchy formula for a disc.*

Problem 32. Let $A = M_n$ where $n \geq 2$ and let $J \in A$ be a Jordan cell, with the value z on the diagonal, i.e.,

$$J = \begin{pmatrix} z & 1 & 0 & \dots & 0 & 0 \\ 0 & z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z & 1 \\ 0 & 0 & 0 & \dots & 0 & z \end{pmatrix}.$$

(1) Show that $\sigma(J) = \{z\}$.
(2) Show that

$$(\lambda I - J)^{-1} = \begin{pmatrix} \frac{1}{\lambda - z} & \frac{1}{(\lambda-z)^2} & \dots & \frac{1}{(\lambda-z)^n} \\ 0 & \frac{1}{\lambda-z} & \dots & \frac{1}{(\lambda-z)^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda-z} \end{pmatrix} \qquad \text{for } \lambda \in \mathbb{C} \setminus \{z\}.$$

(3) Let f be a function holomorphic on a neigborhood of z. Show that

$$\tilde{f}(J) = \begin{pmatrix} f(z) & f'(z) & \frac{f''(z)}{2} & \dots & \frac{f^{(n-1)}(z)}{(n-1)!} \\ 0 & f(z) & f'(z) & \dots & \frac{f^{(n-2)}(z)}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f(z) \end{pmatrix}$$

(4) Deduce that in this case the value of $\tilde{f}(J)$ is not determined by $f|_{\sigma(J)}$.

Hint: *(2) Consider a sufficiently small circle with center a and use the Cauchy formula for higher derivatives.*

Problem 33. Let $A = M_n$ and $E \in A$ be an arbitrary matrix. Let f be a function holomorphic on a neighborhood of $\sigma(E)$.

(1) Express $\tilde{f}(E)$ using the Jordan canonical form of E.
(2) Characterize those matrices E for which $\tilde{f}(E)$ is determined by $f|_{\sigma(E)}$.

Problem 34. Let $A = \ell^1(\mathbb{Z}_2)$ or $A = \ell_1(\mathbb{Z}_3)$. For $x \in A$ and f holomorphic on a neighborhood of $\sigma(x)$ compute the value of $\tilde{f}(x)$.

Problem 35. Let A be a unital Banach algebra and $x \in A$ be an element satisfying one of the following conditions:

(1) $x^n = 0$ for some $n \in \mathbb{N}$;
(2) $x^2 = x$;
(3) $x^2 = -x$;
(4) $x^2 = e$;
(5) $x^3 = x$, but none of the conditions (2)–(4) holds.

Let f be a function holomorphic on a neighborhood of $\sigma(x)$. Compute $\tilde{f}(x)$. In which cases it is determined by $f|_{\sigma(x)}$?

Problem 36. Let $A = \mathcal{C}(K)$, let $g \in A$ and let F be a function holomorphic on a neighborhood of $\sigma(g) = g(K)$. Show that $\tilde{F}(g) = F \circ g$.

Problem 37. Let $A = \ell^1(\mathbb{Z})$ (cf. Problem 12) and $n \in \mathbb{Z}$, $n \neq 0$. By Problem 29 we know that $\sigma(\boldsymbol{e}_n) = \mathbb{T}$. Let g be a function holomorphic on a neighborhood of $\mathbb{T}$. Show that

$$\tilde{g}(\boldsymbol{e}_n) = \sum_{k\in\mathbb{Z}} a_k e_{kn},$$

where $(a_k)_{k\in\mathbb{Z}}$ are the coeficients of the Laurent expansion of g in a neighborhood of $\mathbb{T}$.

Hint: *One can use either the definitions and the formula from Problem 29, or one can prove an analogue of the statement in Problem 30 for Laurent series.*

Problems to section X.4 – ideals, multiplicative functionals and the Gelfand transform

Problem 38. Let $A = \mathcal{C}(K)$.

(1) Let I be a proper closed ideal v A. Show that there exists a nonempty closed set $F \subset K$ such that
$$I = \{f \in \mathcal{C}(K); f|_F = 0\}.$$
(2) Show that maximal ideals in A are exactly subspaces of the form
$$I = \{f \in \mathcal{C}(K); f(x) = 0\},$$
where $x \in K$.
(3) Deduce that the multiplicative functionals on A are exactly the functionals of the form $f \mapsto f(x)$, where $x \in K$.
(4) Explain and prove the statement, that the Gelfand transform of the algebra A is the identity mapping.

`Hint:` *(1) Set $F = \{x \in K; \forall f \in I : f(x) = 0\}$. Show that $F \neq \emptyset$ (otherwise using compactness and the definition of an ideal show that $1 \in I$). Similarly show that for each closed set H disjoint with F there exists a nonnegative $f \in I$ strictly positive on H. Using the Tietze theorem and definition of an ideal further show that there exists $g \in I$ with values in $[0, 1]$ which equals 1 on H. Finally deduce that any function which is zero on F may be approximated by functions from I.*

Problem 39. Let $A = M_n$, where $n \geq 2$. Show that the only proper two-sided ideal in A is the zero ideal.

`Hint:` *Assume that I is a nonzero ideal. Show that it contains at least one matrix with exacly one nonzero entry, then deduce that it contains all such matrices.*

Problem 40. Let $A = \ell^1(G)$, where $(G, +)$ is a commutative group (see Problem 12). Recall that then $A^* = \ell^\infty(G)$. Show that $\varphi \in \ell^\infty(G)$ belongs to $\Delta(A)$ if and only if it is a group homomorphism to the unit circle (i.e., $\varphi : G \to \mathbb{T}$ and $\varphi(g_1 + g_2) = \varphi(g_1)\varphi(g_2)$ for $g_1, g_2 \in G$).

`Hint:` *Observe that $e_{g_1} * e_{g_2} = e_{g_1+g_2}$. To show that the values belong to $\mathbb{T}$ use boundedness of φ and group operations.*

Problem 41. Let $A = \ell^1(\mathbb{Z})$.

(1) Describe $\Delta(A)$ and explain how to understand the equality $\Delta(A) = \mathbb{T}$.
(2) Describe the Gelfand transform of A and (using it) express the spectrum of a general element of A.
(3) Is the Gelfand transform one-to-one? If yes, what is its inverse?
(4) What is the range of the Gelfand transform? Is it onto?

`Hint:` *(1) Use Problem 40 and consider the mapping $\Delta(A) \ni \varphi \mapsto \varphi(1)$. (3) Use the knowledge of Fourier series. (4) Not every continuous function has an absolutely convergent Fourier series.*

Problem 42. Let $A = \ell^1(\mathbb{Z}_n)$, where $n \in \mathbb{N}$, $n \geq 2$.

(1) Describe $\Delta(A)$ and show that it has exactly n elements.
(2) Describe the Gelfand transform A and (using it) express the spectrum of a general element of A.
(3) Is the Gelfand transform one-to-one? If yes, what is its inverse?
(4) What is the range of the Gelfand transform? Is it onto?

`Hint:` *(1) Use Problem 40 and consider the mapping $\Delta(A) \ni \varphi \mapsto \varphi(1)$. (3,4) Use (among others) properties of finite-dimensional spaces.*

Problem 43. Let $A = L^1(\mathbb{R}^n)$ (se Problem 14). Recall that $A^* = L^\infty(\mathbb{R}^n)$. Moreover,

$$\Delta(A) = \{\varphi : \mathbb{R}^n \to \mathbb{T}; \varphi \text{ is continuous and } \forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n : \varphi(\boldsymbol{x} + \boldsymbol{y}) = \varphi(\boldsymbol{x}) \cdot \varphi(\boldsymbol{y})\},$$

which is a nontrivial known fact.

(1) Show that the elements $\Delta(A)$ are exactly the functions $\boldsymbol{x} \mapsto e^{i\langle \boldsymbol{x}, \boldsymbol{y} \rangle}$, where $\boldsymbol{y} \in \mathbb{R}^n$.
(2) Explain what the equality $\Delta(A) = \mathbb{R}^n$ means.
(3) Describe the Gelfand transform of A and explain its relationship to the Fourier transform.

Problem 44. Consider $\mathbb{T}$ as a compact group (the operation is the multiplication), i.e., $\mathbb{T} = \{e^{it}; t \in [0, 2\pi)\} = \{e^{it}; t \in \mathbb{R}\}$. Let $A = L^1(\mathbb{T})$ be equipped with the standard norm and the convolution as a multiplication, i.e.,

$$\|f\| = \frac{1}{\pi} \int_0^{2\pi} \left|f(e^{it})\right| \,\mathrm{d}t, \quad (f * g)(e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{is}) g(e^{i(t-s)}) \,\mathrm{d}s.$$

(1) Show that A is a commutative Banach algebra with no unit.
(2) Using the representation $A^* = L^\infty(\mathbb{T})$ and the known nontrivial fact that

$$\Delta(A) = \{\varphi : \mathbb{T} \to \mathbb{T}; \varphi \text{ is continuous and } \forall x, y \in \mathbb{T} : \varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)\},$$

show that the elements $\Delta(A)$ are exactly the functions $x \mapsto x^n$ for $n \in \mathbb{Z}$ (resp. $e^{it} \mapsto e^{int}$).
(3) Explain what the equality $\Delta(A) = \mathbb{Z}$ means.
(4) Describe the Gelfand transform of A and explain its relationship to the Fourier series.

`Hint:` *(1) Show that it is just a different description of the algebra from Problem 15.*

Problem 45. Let A be a unital commutative Banach algebra of finite dimension. Show that for each $x \in A$ its spectrum $\sigma(x)$ is a finite set with at most $\dim A$ elements.

`Hint:` *Let $x \in A$. It follows that there exists $k \leq n$, such that the element x^k is a linear combination of the elements $1, x, \ldots, x^{k-1}$. Hence, for any $\varphi \in \Delta(A)$ the value $\varphi(x)$ must be a root of a certain polynomial of degree k. Moreover, $\sigma(x) = \{\varphi(x); \varphi \in \Delta(A)\}$ (see Theorem X.25(e)).*