XI. C*-algebras and continuous function calculus

Remark: This chapter is a continuation of the preceding one, all spaces are again assumed to be complex.

XI.1 Algebras with involution and C*-algebras – basic properties

Definition. Let A be a Banach algebra.

- An **involution** on A is a mapping $x \mapsto x^*$ of A into itself such that for each $x, y \in A$ and $\lambda \in \mathbb{C}$ one has

$$(x+y)^* = x^* + y^*, \qquad (\lambda x)^* = \overline{\lambda} x^*, \qquad (xy)^* = y^* x^* \qquad \text{and} \qquad x^{**} = x.$$

- A Banach algebra A with involution is called a C^*-**algebra** if for each $x \in A$ one has
$$\|x^* x\| = \|x\|^2.$$
- If A is a Banach algebra with involution and $x \in A$, the element x is called **selfadjoint** (or **hermitien**) if $x^* = x$; x is called **normal** if $x^* x = x x^*$.

Remarks.

(1) Let A be a Banach algebra with involution. Then $e \in A$ is a left unit if and only if e^* is a right unit. Hence, if A has either a left unit or a right unit, it is unital and the unit is selfadjoint.

(2) If A is a Banach algebra with involution such that

$$\|x^* x\| \geq \|x\|^2 \text{ for } x \in A,$$

then A is a C^*-algebra.

(3) Let A be a C^*-algebra. Then $x \mapsto x^*$ is a conjugate linear isometry of A onto A. Hence,

$$\|x^* x\| = \|x x^*\| = \|x\|^2 = \|x^*\|^2 \text{ for } x \in A.$$

(4) Let A be a nontrivial C^*-algebra with unit e. Then $\|e\| = 1$.

Examples 1.

(1) *The complex field is a commutative C^*-algebra, if the involution is defined by $\lambda^* = \overline{\lambda}$ for $\lambda \in \mathbb{C}$.*

(2) *The algebra $\mathcal{C}_0(T)$ (where T is locally compact space) is a commutative C^*-algebra, if the involution is defined by $f^*(t) = \overline{f(t)}$ for $t \in T$.*

(3) *The matrix algebra M_n is a C^*-algebra if the involution is defined by*

$$\left((a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}\right)^* = (\overline{a_{ji}})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}.$$

(4) *If H is a Hilbert space, then the algebras $L(H)$ and $K(H)$ are C^*-algebras, if the involution T^* is defined to be the adjoint operator to T.*

(5) *On the algebra $L^1(\mathbb{R}^n)$ one can define an involution by $f^*(x) = \overline{f(x)}$, $x \in \mathbb{R}^n$; or by $f^*(x) = \overline{f(-x)}$, $x \in \mathbb{R}^n$. $L^1(\mathbb{R}^n)$ is not a C^*-algebra with any of these involutions.*

Proposition 2 (properties of algebras with involution). *Let A be a Banach algebra with involution and let $x \in A$. Then:*

(a) *Elements $x + x^*$, $i(x - x^*)$, x^*x are selfadjoint.*

(b) *There exist uniquely determined selfadjoint elements $u, v \in A$ such that $x = u + iv$. Moreover, x is normal if and only if $uv = vu$.*

(c) *If A is unital, then $x \in G(A)$ if and only if $x^* \in G(A)$ (in this case $(x^*)^{-1} = (x^{-1})^*$).*

(d) $\sigma(x^*) = \{\overline{\lambda} : \lambda \in \sigma(x)\}$.

Proposition 3 (on the spectral radius and the norm of a normal element). *If A is a C^*-algebra and $a \in A$ is normal, then $r(a) = \|a\|$.*

Corollary 4. *Let A be an algebra with involution. Then there is at most one norm $\|\cdot\|$ such that $(A, \|\cdot\|)$ is a C^*-algebra.*

Proposition 5 (adding a unit). *Let A be a Banach algebra with involution.*

(a) *A^+ is again a Banach algebra with involution, provided the involution is defined by $(a, \lambda)^* = (a^*, \overline{\lambda})$ for $(a, \lambda) \in A^+$.*

(b) *If A is a C^*-algebra, then A^+ is also a C^*-algebra, if the involution is defined as in (a) and the norm on A^+ is defined by*

$$\|(a, \lambda)\| = \max\{|\lambda|, \sup\{\|ab + \lambda b\|; b \in A, \|b\| \le 1\}\}.$$

(c) *If A is a C^*-algebra with no unit, then the norm defined in (b) can be expressed as*

$$\|(a, \lambda)\| = \sup\{\|ab + \lambda b\|; b \in A, \|b\| \le 1\}.$$

Remark: The norm on A^+ defined in Proposition 5(b) differs from the norm given in Proposition X.2(b). It follows from Corollary 4 that the formula from Proposition 5(b) is the unique possible.