Remarks on possible definitions of the resolvent set

$\rho_a(T) = \{\lambda \in \mathbb{C};\ \lambda I - T$ is one-to-one, onto, $(\lambda I - T)^{-1} \in \mathcal{L}(X)\}$

$\rho_b(T) = \{\lambda \in \mathbb{C};\ \lambda I - T$ is one-to-one, onto$\}$

$\rho_c(T) = \{\lambda \in \mathbb{C};\ \lambda I - T$ is one-to-one, $R(\lambda I - T)$ is dense, $(\lambda I - T)^{-1}$ is continuous$\}$

- T closed $\Rightarrow \rho_a(T) = \rho_b(T) = \rho_c(T)$ by Proposition 13

  [T closed, $\lambda \in \mathbb{C} \Rightarrow \lambda I - T$ closed by Prop. 11 (a)]

- T not closed $\Rightarrow \rho_a(T) = \emptyset$

  [$(\lambda I - T)^{-1} \in \mathcal{L}(X) \Rightarrow (\lambda I - T)^{-1}$ closed $\Rightarrow \lambda I - T$ closed
  ↑ Prop. 10 (c)

  Prop. 11 (a)
  $\Rightarrow T$ closed]

- T not closed but with a closed extension

  $\Rightarrow \rho_c(T) = \rho_c(\overline{T}) = \rho_a(\overline{T}) = \rho_b(\overline{T})$

  [by the last item in Remark at the end of Section XII.2]

- $\overline{\rho_b(T)} \cap \rho(\overline{T}) = \emptyset$

  [$\lambda I - \overline{T} = \overline{\lambda I - T}$. By the second item in Remark at the end of Section XII.2: $\lambda \in \rho_b(T) \Rightarrow$

  $\lambda I - T$ is one-to-one, onto $\Rightarrow \overline{\lambda I - T}$ is not one-to-one

  Hence $\lambda I - \overline{T}$ is not one-to-one $\Rightarrow \lambda \notin \rho(\overline{T})$]