Adding a unit to an algebra

Let A be an algebra.

(a) $A^+ = A \times \mathbb{C}$

$$(x,\lambda)\cdot(y,\mu) = (x\cdot y + \lambda y + \mu x, \lambda\mu)$$

Then A^+ is an algebra:

- $(x,\lambda)((y,\mu)\cdot(z,\theta)) = (x,\lambda)\cdot(yz+\mu z+\theta y, \mu\theta) =$
  $= (xyz + \mu xz + \theta xy + \lambda yz + \lambda\mu z + \lambda\theta y + \mu\theta x, \lambda\mu\theta)$

  $((x,\lambda)(y,\mu))\cdot(z,\theta) = (xy+\lambda y+\mu x, \lambda\mu)(z,\theta) =$
  $= (xyz + \lambda yz + \mu xz + \theta xy + \theta\lambda y + \theta\mu x + \lambda\mu z, \lambda\mu\theta)$

  It is the same

- $(x,\lambda)((y,\mu)+(z,\theta)) = (x,\lambda)(y+z, \mu+\theta) =$
  $= (x(y+z) + (\mu+\theta)x + \lambda(y+z), \lambda(\mu+\theta))$
  $= (xy+\mu x+\lambda y, \lambda\mu) + (xz+\theta x+\lambda z, \lambda\theta) =$
  $= (x,\lambda)(y,\mu) + (x,\lambda)(z,\theta)$

- $((x,\lambda)+(y,\mu))\cdot(z,\theta) = (x+y, \lambda+\mu)(z,\theta)$
  $= ((x+y)z + \theta(x+y) + (\lambda+\mu)z, (\lambda+\mu)\theta)$
  $= (xz+\theta x+\lambda z, \lambda\theta) + (yz+\theta y+\mu z, \mu\theta) =$
  $= (x,\lambda)\cdot(z,\theta) + (y,\mu)(z,\theta)$

- $\alpha\cdot((x,\lambda)\cdot(y,\mu)) = \alpha(xy+\lambda y+\mu x, \lambda\mu) = (\alpha xy + \alpha\lambda y + \alpha\mu x, \alpha\lambda\mu)$
  $(\alpha(x,\lambda))(y,\mu) = (\alpha x, \alpha\lambda)\cdot(y,\mu) = (\alpha xy + \mu\alpha x + \alpha\lambda y, \alpha\lambda\mu)$
  $(x,\lambda)\cdot(\alpha(y,\mu)) = (x,\lambda)(\alpha y, \alpha\mu) = (\alpha xy + \alpha\mu x + \lambda\alpha y, \lambda\alpha\mu)$

Moreover, A^+ commutative $\iff$ A commutative

$A \subset A^+$: $x \mapsto (x, 0)$ is an algebraic isomorphism (into) $\quad \big\ulcorner (x,0)\cdot(y,0) = (xy, 0) \big\lrcorner$

$(0, 1)$ is a unit of A^+

$$(0,1)\cdot(x,\lambda) = (0\cdot x + x + \lambda\cdot 0, 1\lambda) = (x, \lambda)$$
$$(x,\lambda)\cdot(0,1) = (x\cdot 0 + 1\cdot x + \lambda\cdot 0, \lambda\cdot 1) = (x,\lambda)$$

Moreover, it is a unique possibility:
If $B \supset A$ is an algebra with unit $e \in B \setminus A$, then

$\varphi : A^+ \to B \quad \varphi(a,\lambda) = a + \lambda e$ is an isomorphism onto $\operatorname{span}(A \cup \{e\})$

- φ is a linear bijection (clear)
- $\varphi((a,\lambda)(b,\mu)) = \varphi(ab + \lambda b + \mu a, \lambda\mu) = ab + \lambda b + \mu a + \lambda\mu e$

$\varphi(a,\lambda)\,\varphi(b,\mu) = (a+\lambda e)(b+\mu e) = ab + \lambda b + \mu a + \lambda\mu e$

(b) Assume that A is a Banach algebra:

Define: $\|(x,\lambda)\| = \|x\| + |\lambda| \implies A^+$ is a Banach algebra with unit $(0,1)$

$\|(0,1)\| = 1$

A complete $\implies A^+$ complete $\quad (A^+ = A \oplus_{\ell_1} \mathbb{C})$

$$\|(x,\lambda)(y,\mu)\| = \|xy + \lambda y + \mu x, \lambda\mu)\| =$$

$$= \|xy + \lambda y + \mu x\| + |\lambda\mu| \le \|xy\| + |\lambda|\,\|y\| + |\mu|\,\|x\| + |\lambda|\cdot|\mu|$$

$$\le \|x\|\cdot\|y\| + |\lambda|\,\|y\| + |\mu|\cdot\|x\| + |\lambda|\cdot|\mu| = (\|x\| + |\lambda|)(\|y\| + |\mu|)$$

$$= \|(x,\lambda)\|\cdot\|(y,\mu)\|$$