Proof of Theorem VI.21 Let T be a normal operator on H

(1) Let $B, C \in \mathcal{L}(H)$ be as in Lemma 19
Then $BT \subset TB$ & $BC = CB$

$\Big[\ BT = BT(I+T^*T)B = B(T+TT^*T)B =$

$(I+T^*T)B = I$

$\supset$ by Prop. V.14 (iii)

Moreover, $D(T(I+T^*T)) =$
$= \{x \in D(T^*T),\ x + T^*Tx \in D(T)\} =$
$= \{x \in D(T^*T),\ T^*Tx \in D(T)\} =$
$= D(TT^*T) = D(TT^*T) \cap D(T)$

$T^*T = TT^*$

$= B(T + T^*TT)B = \underbrace{B(I+T^*T)}_{\subset I} TB \subset TB$

we already know $BT \subset TB$

$BC = BTB \subset TBB = CB$

But $BC, CB \in \mathcal{L}(H)$, so $BC = CB$ $\Big]$

(2) Recall: $B \geq 0$, $\|B\| \leq 1 \Rightarrow \sigma(B) \subset [0,1]$,
0 is not an eigenvalue (B is one-to-one), so $E_B(\{0\}) = 0$

Set $P_j := \chi_{(\frac{1}{j+1}, \frac{1}{j}]}(B)$ (measurable calculus)

$S_j := (\frac{1}{\varphi} \chi_{(\frac{1}{j+1}, \frac{1}{j}]})(B)$, where $\varphi(t) = t$

Then $P_j, S_j \in \mathcal{L}(H)$, commute with each other and with B, P_j are OG projections

P_j mutually orthogonal

all these operators commute with C (as $CB = BC$)

$P_j = S_j B = B S_j$ ($B = \widetilde{id}(B) = \widetilde{\varphi}(B)$)

③ $\sum_{j=1}^{\infty} P_j = I$

$\ulcorner$ $\left\langle \left(\sum_{j=1}^{\infty} P_j\right) x, y \right\rangle = \lim_{n\to\infty} \left\langle \left(\sum_{j=1}^{n} P_j\right) x, y \right\rangle =$

$= \lim_{n\to\infty} \langle \widetilde{\chi_{(\frac{1}{n+1}, 1]}}(B) x, y \rangle = \langle \widetilde{\chi_{(0,1]}} x, y \rangle =$

$= \langle \tilde{1} x, y \rangle = \langle I x, y \rangle = \langle x, y \rangle$

$\chi_{(0,1]} = 1$ E_B - a.e., as $\sigma(B) \subset [0,1]$, $E_B(\{0\}) = 0$ $\lrcorner$

④ $TP_j \in \mathcal{L}(H)$, $P_j T \subset T P_j$

$\ulcorner$ $TP_j = TBS_j = CS_j \in \mathcal{L}(H)$

$P_j T = S_j BT \underset{①}{\subset} S_j TB = S_j C = CS_j = TP_j$ $\lrcorner$

⑤ TP_j is a normal operator

$\ulcorner$ $(TP_j)^* \overset{④}{\subset} (P_j T)^* = T^* P_j \Rightarrow (TP_j)^* = T^* P_j$

Prop. 11(c) as $TP_j \in \mathcal{L}(H)$

So, $\forall x \in H$: $\|(TP_j)^* x\| = \|T^* P_j x\| = \|TP_j x\|$

$\uparrow$ L39(b), $P_j x \in D(T)$

as $TP_j \in \mathcal{L}(H)$

So, TP_j is normal, by a proposition from IX.1 $\lrcorner$

⑥ Let E_j be the spectral measure of TP_j ($\mathcal{A}_j$ the σ-algebra)

Then $E_j(A)$ commutes with P_j, $A \in \mathcal{A}_j$

$\ulcorner$ enough to observe: P_j commutes with TP_j

$P_j TP_j \underset{④}{\subset} TP_j P_j$ ~~that~~ Since P_j and $TP_j \in \mathcal{L}(H)$,

necessarily $P_j TP_j = TP_j P_j$ $\lrcorner$

7 Let $E := \sum_{j=1}^{\infty} E_j \cdot P_j$, i.e.

$$E(A) = \sum_{j=1}^{\infty} E_j(A) P_j, \quad A \in \mathcal{A} = \bigcap_{j=1}^{\infty} \mathcal{A}_j$$

Then E is a well-defined spectral measure

- $E_j(A) P_j = P_j E_j(A) \Rightarrow E_j(A) P_j$ is an OS projection

~~so~~
Moreover, as P_j are mutually OS, also these are mutually OS, hence their sum is an OS projection.

So (i) and (ii) hold

(iii): $E(\emptyset) = 0$ - clear

$E(\mathfrak{X}) = I$: $E(\mathfrak{X}) = \sum_j E_j(\mathfrak{X}) P_j = \sum_j P_j = I$ by 3

(iv) .. clear

(v) $E(A \cap B) = \sum_j E_j(A \cap B) P_j = \sum_j E_j(A) E_j(B) P_j$

$$E(A) E(B) = E(A) \sum_j E_j(B) P_j = \sum_j E(A) P_j E_j(B) =$$

$$= \sum_j E_j(A) P_j E_j(B) = \sum_j E_j(A) E_j(B) P_j$$

$$E(A) P_j = \left(\sum_k E_k(A) P_k\right) P_j = E_j(A) P_j$$

(vi) clear

(vii) $E_{x,x}(A) = \langle E(A)x, x \rangle = \sum_j \langle E_j(A) P_j x, x \rangle =$

$$= \sum_j \langle E_j(A) P_j x, P_j x \rangle = \sum_j E_{P_j x, P_j x}(A)$$

$$\Rightarrow E_{x,x} = \sum_j E_{P_j x, P_j x}$$, so it will work.

(8) $T = \int \mathrm{id}\, dE$. By L 3 a (c) it is enough to show "$\subset$"

$$x \in D(T) \Rightarrow \int |z|^2 \, dE_{x,x} = \sum_j \int |z|^2 \, d(E_j)_{P_j x, P_j x} =$$

$$= \sum_j \|TP_j x\|^2 = \sum_j \|P_j T x\|^2 \underset{(3)}{=} \|Tx\|^2 \Rightarrow x \in D\left(\int \mathrm{id}\, dE\right)$$

Thm 8 (d) $\qquad$ $x \in D(T)$, $P_j T \subset T P_j$

$x, y \in D(T)$:

$$\left\langle \left(\int \mathrm{id}\, dE\right) x, y \right\rangle = \int \mathrm{id}\, dE_{x,y} = \sum_j \int \mathrm{id}\, d(E_j)_{P_j x, P_j y}$$

$$= \sum_j \langle \cancel{P_j T x, y} \ \cancel{\langle P} \ TP_j x, P_j y \rangle = \sum_j \langle TP_j x, y \rangle =$$

$$\underset{(3)}{=} \langle Tx, y \rangle$$

(9) Uniqueness: Let $T = \int \mathrm{id}\, dE$

$$\Rightarrow I + T^* T = \int (1 + |z|^2)\, dE \qquad (\text{Thm } 12 \text{ (c)})$$

$$\Rightarrow B = \int \frac{1}{1+|z|^2}\, dE \qquad (\text{Thm } 12 \text{ (d)})$$

$$C = \int \frac{z}{1+|z|^2}\, dE \qquad (\text{Thm } 12 \text{ (5)})$$

Set $A_j = \{ z \in \mathbb{C} \ ; \ \frac{1}{1+|z|^2} \in (\frac{1}{j+1}, \frac{1}{j}] \}$

$$\Rightarrow P_j = \int \chi_{A_j}\, dE \qquad (\text{using Lemma } 14)$$

$$TP_j = \int z \chi_{A_j}(z)\, dE \qquad (\text{Thm } 12 \text{ (5)})$$

Lemma 9

$\Rightarrow$ E_j (the spectral measure of TP_j) is the image of E by $z \mapsto z\chi_{A_j}(z)$

So, $E_j(A) = E(\{z ; z\chi_{A_j}(z) \in A\}) = \begin{cases} E(A \cap A_j), & 0 \notin A \\ E((A \cap A_j) \cup (\mathbb{C} \setminus A_j)) & \text{if } 0 \in A \end{cases}$

So, $E_j(A)P_j = E(A \cap A_j)$, hence $E(A) = \sum_j E_j(A)P_j$.

So, the definition from (7) is the unique possible

<u>Corollary VI.22</u> T normal. Then T bdd $\Leftrightarrow$ $\sigma(T)$ bdd

<u>Proof</u>: The same as Corollary 18, just use Theorem 21

<u>Corollary VI.23</u> $T = \int f \, dE$ $\Rightarrow$ the spectral measure of T is the image of E by f

Proof: $F := f(E)$ in the sense of L 16

By L 16 $\int id \, dF = \int f \, dE = T$

So, by the uniqueness part of Thm 21, F is the spectral measure of T $\square$