Spectral measure, measurable calculus:

① Let H be a Hilbert space and $T \in \mathcal{L}(H)$ a normal operator.
Let $f \mapsto \tilde{f}(T)$, $f \in \mathcal{C}(\sigma(T))$, be the cts functional calculus

Fix $x, y \in H$. Then $f \mapsto \langle \tilde{f}(T)x, y \rangle$ is a linear functional on $\mathcal{C}(\sigma(T))$. Moreover,

$$|\langle \tilde{f}(T)x, y\rangle| \leq \|\tilde{f}(T)\| \, \|x\| \, \|y\| = \|f\|_\infty \|x\| \, \|y\|$$

So, the norm of this functional is $\leq \|x\| \cdot \|y\|$

Hence, by the Riesz representation theorem $\exists!\, E_{x,y}$, a complex Borel measure on $\sigma(T)$ s.t.

$$\langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y}, \quad f \in \mathcal{C}(\sigma(T))$$

Moreover, $\|E_{x,y}\| \leq \|x\| \cdot \|y\|$. (This proves Prop. 2 (d))

② $x \mapsto E_{x,y}$ is linear for $y \in H$, $y \mapsto E_{x,y}$ is conjugate linear for $x \in H$

$f \in \mathcal{C}(\sigma(T))$

$$\int_{\sigma(T)} f \, dE_{x_1+x_2, y} = \langle \tilde{f}(T)(x_1+x_2), y\rangle = \langle \tilde{f}(T)x_1, y\rangle + \langle \tilde{f}(T)x_2, y\rangle =$$

$$= \int_{\sigma(T)} f \, dE_{x_1, y} + \int_{\sigma(T)} f \, dE_{x_2, y} = \int_{\sigma(T)} f \, d(E_{x_1,y} + E_{x_2,y})$$

$$\int_{\sigma(T)} f \, dE_{\alpha x, y} = \langle \tilde{f}(T)(\alpha x), y \rangle = \alpha \langle \tilde{f}(T)x, y\rangle = \alpha \int_{\sigma(T)} f \, dE_{x,y} = \int_{\sigma(T)} f \, d(\alpha E_{x,y})$$

So, $x \mapsto E_{x,y}$ is linear. The case $y \mapsto E_{x,y}$ is similar

[This proves Prop. 2 (a, b)]

(3) $x \in H \Rightarrow E_{x,x}$ is a nonnegative measure

⌈ To prove this, it is enough to show

$$f \in \mathcal{C}(\sigma(T)),\ f \geq 0 \Rightarrow \int_{\sigma(T)} f \, dE_{x,x} \geq 0 \quad \text{(by the Riesz theorem)}$$

So, fix $f \in \mathcal{C}(\sigma(T))$, $f \geq 0$. Then $\sigma(\hat{f}(T)) = f(\sigma(T)) \subset [0,\infty)$.
Moreover, since f is real-valued, i.e. $\bar{f} = f$, ~~Ø~~ we get that $\hat{f}(T)$ is self-adjoint. Thus $\hat{f}(T)$ is a positive operator, so $\langle \hat{f}(T)x, x\rangle \geq 0$ for $x \in H$ (Prop. 7 (c)) ⌋

So, Prop. 2 (c) is proved.

(4) $$E_{x,y} = \frac{1}{4}\left(E_{x+y,x+y} - E_{x-y,x-y} + iE_{x+iy,x+iy} - iE_{x-iy,x-iy}\right),$$

$x, y \in H$

(This is Prop. 2 (e))

⌈ This can be proved by a direct computation using

first (a, b) (i.e. (2) above).

See also the proof of Lemma V.3. ⌋

(5) Let $\mathcal{A}$ denote the σ-algebra of all the subsets of $\sigma(T)$ which are $E_{x,y}$ - measurable for each $x,y \in H$

[recall that A is $E_{x,y}$-measurable if there are Borel sets B, C s.t. $B \subset A \subset C$ and $|E_{x,y}|(C \setminus B) = 0$]

Note that $A \in \mathcal{A} \Leftrightarrow \forall x \in H$ A is $E_{x,x}$ -measurable

($\Rightarrow$ obvious $\Leftarrow$ by Prop. 2: (e), see (4) above)

(6) Let $f : \sigma(T) \to \mathbb{C}$ be a bdd $\mathcal{A}$-measurable function.

Then $B_f(x,y) = \int_{\sigma(T)} f \, dE_{x,y}$ satisfies the assumptions of

Prop. 1: (the first two properties follow from Prop. 2: (a,b),
and for $x,y \in B_H$ $|B_f(x,y)| = |\int_{\sigma(T)} f \, dE_{x,y}| \leq \int_{\sigma(T)} |f| \, d|E_{x,y}| \leq$

$\leq \|f\|_\infty \|x\| \cdot \|y\|$, so $\|B_f\| \leq \|f\|_\infty$)

Thus by Prop. 1: $\exists! \, \tilde{f}(T) \in L(H)$ s.t.

$$\langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y}, \quad x,y \in H$$

If f is cts, then this $\tilde{f}(T)$ coincides with the result of the cts functional calculus (by uniqueness)

The assignement $f \mapsto \tilde{f}(T)$ is called the <u>measurable calculus for T</u>.

(7) $A \in \mathcal{A} \Rightarrow E_T(A) := \widetilde{\chi_A}(T)$

E_T is the <u>spectral measure of T</u>

(8) $\mathcal{N} = \{ A \in \mathcal{A} \,;\ \forall x,y \in H \ |E_{x,y}|(A) = 0 \}$

Then $\mathcal{N} = \{ A \in \mathcal{A} \,;\ \forall x \in H : E_{x,x}(A) = 0 \}$

($\subset$ obvious $\supset$ by Prop. 12 (d))

(9) f, g bdd $\mathcal{A}$-measurable, $\{ \lambda \in \sigma(T) \,;\ f(\lambda) \neq g(\lambda) \} \in \mathcal{N}$
$\Rightarrow \tilde{f}(T) = \tilde{g}(T)$

⌈ $x,y \in H \Rightarrow \quad f = g \quad |E_{x,y}|$-a.e.,

so $\quad \langle \tilde{f}(T)x, y \rangle = \int_{\sigma(T)} f \, dE_{x,y} = \int_{\sigma(T)} g \, dE_{x,y} = \langle \tilde{g}(T)x, y \rangle$ ⌋

(10) Let $L^{\infty}(E_T)$ denote the space of equivalence classes of bdd $\mathcal{A}$-measurable functions, where f, g are equivalent iff $\{ \lambda \,;\ f(\lambda) \neq g(\lambda) \} \in \mathcal{N}$

$[f] \in L^{\infty}(E_T)$ ... define $\| [f] \| := \operatorname{ess\,sup}_{\sigma(T)} |f| =$

$= \inf \{ c > 0 \,;\ \{ \lambda \in \sigma(T) \,;\ |f(\lambda)| > c \} \in \mathcal{N} \}$

Then $L^{\infty}(E_T)$ is a unital commutative C^*-algebra with the natural operations

(11) By (9) we see that the measurable calculus

$f \mapsto \tilde{f}(T)$ can be interpreted as a mapping $L^{\infty}(E_T) \to L(H)$

(12) It follows from Luzin theorem, that:

Let K be a compact metric space, μ a finite Borel measure on K, and $f : K \to \mathbb{C}$ be a bdd μ-measurable function (i.e. f is measurable w.r.t. to the completion of μ)

Then there is a sequence (f_n) of cts functions on K s.t.

- (f_n) is uniformly bdd
- $f_n \to f$ μ-a.e.

In pact., there is $g : K \to \mathbb{C}$ bdd Borel function s.t. $f = g$ μ-a.e.

(13) H separable $\Rightarrow$ $\forall f$ bdd $\mathcal{A}$-measurable

- $\exists (f_n)$ a uniformly bdd sequence of cts functions s.t. $f_n \to f$ except for a set from $\mathcal{N}$
- $\exists g : \sigma(T) \to \mathbb{C}$ bdd Borel s.t. $f = g$ except on a set from $\mathcal{N}$

$\Big[$ $(x_n)_{n \in \mathbb{N}}$ dense in H, not containing 0

Then $\mathcal{N} = \{ A \in \mathcal{A} \, ; \, E_{x_n, x_n}(A) = 0 \text{ for } n \in \mathbb{N} \}$

$\subset$ obvious; $\supset$: $(x,y) \mapsto E_{x,y}(A)$ is cts by Prop. 13
so $x \mapsto E_{x,x}(A)$ is also cts

Let $\mu := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{E_{x_n, x_n}}{\|x_n\|^2}$ $\Rightarrow$ μ is a finite Borel measure on $\sigma(T)$

and $\mathcal{N} =$ μ-null sets, so we can apply (12) to μ $\Big]$