Proof of Proposition V.1

H, K Hilbert spaces, $T \in L(H, K)$

(i) $\Rightarrow$ (ii) Suppose T is unitary, i.e. $T^* = T^{-1}$

Then • T is an isometry

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle x, T^*Tx \rangle \underset{\uparrow \, T^*T = I_H}{=} \langle x, x \rangle = \|x\|^2$$

• T is onto as it has an inverse

(ii) $\Rightarrow$ (iii) T is an onto isometry $\Rightarrow$ T is an into isometry
[trivial]

(iii) $\Rightarrow$ (iv) Suppose T is an isometry of H into K

Then for $x, y \in H$ we have

$$\langle Tx, Ty \rangle_K \underset{\uparrow}{=} \frac{1}{4}\left(\|Tx + Ty\|^2 - \|Tx - Ty\|^2 + i\|Tx + iTy\|^2 - i\|Tx - iTy\|^2\right)$$

polarization identity

$$= \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) = \langle x, y \rangle_H$$

T is an isometry

(iv) $\Rightarrow$ (iii) Suppose $\langle Tx, Ty \rangle_K = \langle x, y \rangle_H$, $x, y \in H$

Apply for $y = x$ and deduce that T is an isometry

(iii) $\Rightarrow$ (ii) T is an into isometry & T is onto $\Rightarrow$ T is an onto isometry
[if T is onto] [trivial]

(ii) $\Rightarrow$ (i) Suppose T is an isometry of H onto K. Then T^{-1} exists.

Moreover, by the already proved (iii) $\Rightarrow$ (iv) we have

$$\langle Tx, Ty \rangle_K = \langle x, y \rangle_H \text{ for } x, y \in H$$

Thus for $x \in H, y \in K$ we have

$$\langle Tx, y \rangle_K = \langle Tx, TT^{-1}y \rangle_K = \langle x, T^{-1}y \rangle_H. \quad \text{Thus } T^{-1} = T^*$$