FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2023/2024

PROBLEMS TO CHAPTER VII

Problems to Section VII.1 – test functions and weak derivatives

Problem 1. Set

$$\mathcal{W}((a,b)) = \{f \in L^1_{\mathrm{loc}}((a,b)); f \text{ has a weak derivative in } L^1_{\mathrm{loc}}((a,b))\}.$$

For $f \in \mathcal{W}((a,b))$ let f' denote the weak derivative of f. Let $p \in [1,\infty]$. Let

$$W^{1,p}((a,b)) = \{f \in \mathcal{W}((a,b)); f \in L^p((a,b)) \text{ and } f' \in L^p((a,b))\}.$$

For $f \in W^{1,p}$ set

$$\|f\|_{1,p} = \begin{cases} (\|f\|_p^p + \|f'\|_p^p)^{1/p}, & \text{if } p < \infty, \\ \max\{\|f\|_\infty, \|f'\|_\infty\}, & \text{if } p = \infty. \end{cases}$$

Show that $(W^{1,p}((a,b)), \|\cdot\|_{1,p})$ is a Banach space and that in case $p = 2$ it is a Hilbert space.

`Hint:` *Show that $f \mapsto (f, f')$ is an isometry $W^{1,p}((a,b))$ onto a closed subspace of $(L^p((a,b)) \times L^p((a,b)), \|\cdot\|_p)$.*

Problem 2. Show that the space $W^{1,1}((0,1))$ is isomorphic to the space $AC([0,1])$ from Problem I.15 (see Introduction to functional analysis, Problem to Chapter I).

`Hint:` *Use Theorem VII.4(b).*

Problem 3. Show that $\mathcal{C}^\infty$-functions on $[a,b]$ form a dense subspace of $W^{1,p}((a,b))$ for each $p \in [1,\infty)$.

`Hint:` *Let $f \in W^{1,p}((a,b))$. Using Lemma VII.1 approximate f' by a test function and take a suitable antiderivative of this test function.*

Problem 4. Show that $\mathscr{D}((0,1))$ is not a dense subspace of $W^{1,1}((0,1))$.

`Hint:` *Consider the constant function equal to* 1.

Problem 5. Show that the function $f(t) = \log|t|$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$, but it has no weak derivative in $L^1_{\mathrm{loc}}(\mathbb{R})$.

`Hint:` *If a function $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ is a weak derivative of f on $\mathbb{R}$, then $g|_{(0,\infty)}$ is a weak derivative of $f|_{(0,\infty)}$ on $(0,\infty)$.*

Problem 6.
(1) Show that each $f \in L^1_{\mathrm{loc}}((a,b))$ is the weak derivative of some continuous function on (a,b).
(2) Show that each signed or complex regular Borel measure on (a,b) is the weak derivative of some right-continuous function on (a,b).
(3) Show that a signed or complex regular Borel measure μ on (a,b) is the weak derivative of some continuous function on (a,b) if and only if $\mu(\{x\}) = 0$ for each $x \in (a,b)$.

Problem 7. Compute weak derivatives of the following function on $\mathbb{R}$. In which cases the weak derivative is again a function and in which cases it is a measure on $\mathbb{R}$?

(1) $f(x) = |x|$, $x \in \mathbb{R}$;
(2) $\chi_{(0,\infty)}$;
(3) $\chi_{(0,1)}$;
(4) the Cantor function.

Problem 8. Find a continuous function on $[0,1]$ such that no measure on $(0,1)$ is its weak derivative.

Problem 9. For each $n \in \mathbb{N}$ find a function $\varphi_n \in \mathscr{D}(\mathbb{R})$ such that $\varphi_n(0) = \varphi_n'(0) = \cdots = \varphi_n^{(n-1)}(0) = 0$ and $\varphi_n^{(n)}(0) \neq 0$.

`Hint:` *Take* $\varphi_n(x) = x^n \psi(x)$ *for a suitable* ψ.

Problem 10. For each multiindex $\alpha \in \mathbb{N}_0^d$ find $\varphi_\alpha \in \mathscr{D}(\mathbb{R}^d)$ such that $D^\beta \varphi_\alpha(0) = 0$ for each $\beta < \alpha$ and $D^\alpha \varphi_\alpha(0) \neq 0$.

`Hint:` *Take* $\varphi_\alpha(\boldsymbol{x}) = \boldsymbol{x}^\alpha \psi(\boldsymbol{x})$ *for a suitable* ψ.

Problem 11. Let $\varphi \in \mathscr{D}(\mathbb{R})$ and $a \in \mathbb{R}$. Show that the function

$$\psi(x) = \begin{cases} \frac{\varphi(x)-\varphi(a)}{x-a}, & x \in \mathbb{R} \setminus \{a\}, \\ \varphi'(a), & x = a \end{cases}$$

belongs to $\mathscr{D}(\mathbb{R})$.

`Hint:` *Show that* $\psi(x) = \int_0^1 \varphi'(a + t(x-a))\,\mathrm{d}t$, $x \in \mathbb{R}$, *and use the theorem on a derivative with respect to a parameter.*

Problem 12. Let $\varphi \in \mathscr{D}(\mathbb{R})$ and $a \in \mathbb{R}$ be such that $\varphi(a) = 0$. Show that there exists some $\psi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(x) = (x-a)\psi(x)$ for $x \in \mathbb{R}$.

`Hint:` *Use the preceding problem.*

Problems to Section VII.2 – distributions and operations with them

Problem 13. Find a sequence $(\varphi_n) \subset \mathscr{D}(\mathbb{R})$ such that for each $k \in \mathbb{N}_0$ we have $\varphi_n^{(k)} \rightrightarrows 0$ on $\mathbb{R}$, but φ_n do not converge to zero in $\mathscr{D}(\mathbb{R})$.

Problem 14. Find a sequence $(\varphi_n) \subset \mathscr{D}((0,1))$ such that for each $k \in \mathbb{N}_0$ we have $\varphi_n^{(k)} \rightrightarrows 0$ on $(0,1)$ but φ_n do not converge to zero in $\mathscr{D}((0,1))$.

Problem 15. For $\varphi \in \mathscr{D}(\mathbb{R})$ define

$$\Lambda_{1/x}(\varphi) = \lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\,\mathrm{d}x \right)$$

(1) Show that $\Lambda_{1/x}$ is a distribution on $\mathbb{R}$.
(2) Show that $\Lambda_{1/x}|_{\mathscr{D}((0,\infty))}$ is of the form Λ_f for some $f \in L^1_{\mathrm{loc}}((0,\infty))$ (i.e., $\Lambda_{1/x}|_{\mathscr{D}((0,\infty))}$ is a regular distribution), but $\Lambda_{1/x}$ is not a regular distribution (i.e., it is not of the form Λ_f for any $f \in L^1_{\mathrm{loc}}(\mathbb{R})$).
(3) Show that $\Lambda_{1/x}$ is a derivative of the regular distribution Λ_g where $g(x) = \log|x|$.
(4) Compute the derivative of the distribution $\Lambda_{1/x}$.
(5) Show that the distribution $\Lambda_{1/x}$ is of order 1 (and not of order 0).

`Hint:` *(5) For a proof that it is of order at most* 1 *use (3). To prove it is not of order* 0 *use the definition and (for example) the compact set* $[0,1]$.

Problem 16. Let $\Lambda_{1/x}$ be the distribution from the preceding problem, let δ_0 be the Dirac measure supported in zero and let $f(x) = x$, $x \in \mathbb{R}$.

(1) Show that $f \cdot \Lambda_{1/x} = \Lambda_1$.
(2) Show that $f \cdot \Lambda_{\delta_0} = 0$.
(3) Show that on $\mathscr{D}'(\mathbb{R})$ it is not possible to define an associative multiplication satisfying $\Lambda_f \cdot U = f \cdot U$ for $f \in \mathcal{C}^\infty(\mathbb{R})$ and $U \in \mathscr{D}'(\mathbb{R})$.

Problem 17. Which of the following formulas define a distribution on $\mathbb{R}$?

(1) $\Lambda(\varphi) = \sum_{n=1}^\infty \varphi(n)$.
(2) $\Lambda(\varphi) = \sum_{n=1}^\infty n\varphi(n)$.
(3) $\Lambda(\varphi) = \sum_{n=1}^\infty n! \cdot \varphi(n)$.
(4) $\Lambda(\varphi) = \sum_{n=1}^\infty \varphi(\frac{1}{n})$.
(5) $\Lambda(\varphi) = \sum_{n=1}^\infty \frac{1}{n}\varphi(\frac{1}{n})$.
(6) $\Lambda(\varphi) = \sum_{n=1}^\infty \frac{1}{n^2}\varphi(\frac{1}{n})$.

Problem 18. Which of the formulas from the preceding problem define a distribution on $(0, \infty)$?

Problem 19. Let $f = \frac{1}{2}\chi_{\{(t,x)\in\mathbb{R}^2; t>|x|\}}$.

(1) Show that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.
(2) Show that $D^{(2,0)}\Lambda_f - D^{(0,2)}\Lambda_f = \Lambda_{\delta_{(0,0)}}$, i.e., „$f$ is a solution of the equation $\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = \delta_{(0,0)}$ in distributions“.

`Hint:` *Use definitions, Fubini theorem and integration by parts.*

Problem 20. Let U be a distribution on (a, b) and $f \in \mathcal{C}^\infty((a, b))$. Show that $(f \cdot U)' = f' \cdot U + f \cdot U'$.

Problem 21. Find a function f which is zero on $(-\infty, 0)$ and $\mathcal{C}^\infty$ on $[0, \infty)$, such that „it solves the equation $y'' + y = \delta_0$ in distributions,“ i.e., such that $(\Lambda_f)'' + \Lambda_f = \Lambda_{\delta_0}$.

`Hint:` *Start by plugging to the equation any $\varphi \in \mathscr{D}((0, \infty))$. Using definitions, integration by parts and Lemma VII.2 show that on $(0, \infty)$ f must solve the differenctial equation $y'' + y = 0$. To determine which solution provides the sought function f plug to the equation the function φ_1 from Problem 9.*

Problem 22. For $\varphi \in \mathscr{D}(\mathbb{R}^2)$ set

$$T(\varphi) = \int_{U(0,1)} \frac{\varphi(\boldsymbol{x}) - \varphi(0)}{\|\boldsymbol{x}\|^2}\,\mathrm{d}\boldsymbol{x} + \int_{\mathbb{R}^2\setminus U(0,1)} \frac{\varphi(\boldsymbol{x})}{\|\boldsymbol{x}\|^2}\,\mathrm{d}\boldsymbol{x}.$$

(1) Show that $T \in \mathscr{D}'(\mathbb{R}^2)$.
(2) Show that $f \cdot T = \Lambda_1$, where $f(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$.

`Hint:` *(1) The second summand is a regular distribution. In the first summand express the difference $\varphi(\boldsymbol{x}) - \varphi(0)$ by an integral of a derivative using Newton-Leibniz formula, use Cauchy-Schwarz inequality and the fact, that the function $\boldsymbol{x} \mapsto \frac{1}{\|x\|}$ is integrable on $U(0, 1)$ (this can be computed using polar coordinates) to prove that it is a distribution of order one.*

Problem 23. Let $\varphi \in \mathscr{D}(\mathbb{R}^d)$. For $r > 0$ define function φ_r by the formula $\varphi_r(\boldsymbol{x}) = \varphi(r\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$.

(1) Show that $\varphi_r \in \mathscr{D}(\mathbb{R}^d)$.
(2) Assuming $r > 1$, show that $\|\varphi\|_N \le \|\varphi_r\|_N \le r^N \|\varphi\|_N$ for each $N \in \mathbb{N}_0$.
(3) Assuming $r \in (0, 1)$, show that $r^N \|\varphi\|_N \le \|\varphi_r\|_N \le \|\varphi\|_N$ for each $N \in \mathbb{N}_0$.

`Hint:` *Use the respective definitions and the theorem on the derivative of a composition.*

Problem 24. Which of the following formulas define a distribution on $\mathbb{R}$?

(1) $\Lambda(\varphi) = \sum_{n=1}^{\infty} \varphi^{(n)}(n)$.
(2) $\Lambda(\varphi) = \sum_{n=1}^{\infty} n\varphi^{(n)}(n)$.
(3) $\Lambda(\varphi) = \sum_{n=1}^{\infty} n! \cdot \varphi^{(n)}(n)$.
(4) $\Lambda(\varphi) = \sum_{n=1}^{\infty} \varphi^{(n)}(\frac{1}{n})$.
(5) $\Lambda(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n^2}\varphi^{(n)}(\frac{1}{n})$.
(6) $\Lambda(\varphi) = \sum_{n=1}^{\infty} \frac{1}{n!}\varphi^{(n)}(\frac{1}{n})$.

`Hint:` *(4)–(6) Use the characterization of distributions in Proposition VII.6(4) and Problems 9 and 23.*

Problem 25. Which of the formulas from the preceding problem define a distribution on $(0,\infty)$? Are they of finite order?

`Hint:` *Use the respective definitions and Problems 9 and 23.*

Problem 26. Compute $D^{(1,0)}\Lambda_f$ and $D^{(0,1)}\Lambda_f$ for the following functions on $\mathbb{R}^2$. Determine whether the resulting distribution is regular or induced by a measure.

(1) $f = \chi_{(0,\infty)\times\mathbb{R}}$;
(2) $f = \chi_{(0,\infty)\times(0,\infty)}$;
(3) $f = \chi_{(0,1)\times(0,1)}$;
(4) $f = \chi_{\{(x,y);y>x\}}$;
(5) $f = \chi_{\{(x,y);y>2x\}}$.

Problem 27. Let $n \in \mathbb{N}$ and let $U \in \mathscr{D}'((a,b))$ satisfy $U^{(n)} = 0$. Show that there exists a polynomial P of degree less than n such that $U = \Lambda_P$.

`Hint:` *Use induction and Proposition VII.9.*

Problem 28. Find all distributions on $\mathbb{R}$ satisfying

(1) $U' = \Lambda_{\delta_0}$;
(2) $U'' = \Lambda_{\delta_0}$.

`Hint:` *Find a particular solution and use the preceding problem.*

Problem 29. Let $a \in \mathbb{R}$ a $f(x) = x - a$, $x \in \mathbb{R}$. Find all distributions on $\mathbb{R}$ satisfying

(1) $fU = 0$;
(2) $f^2U = 0$;
(3) $fU = \Lambda_1$;
(4) $f^2U'' = 0$.

`Hint:` *(1): Use Problem 12 to prove that* $\operatorname{Ker}\Lambda_{\delta_a} \subset \operatorname{Ker} U$.

Problem 30. Consider the function $f(\boldsymbol{x}) = \frac{1}{\|\boldsymbol{x}\|}$ on $\mathbb{R}^3$.

(1) Show that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$.
(2) For $j = 1,2,3$ set $g_j(\boldsymbol{x}) = -\frac{x_j}{\|\boldsymbol{x}\|^3}$. Show that $g_j \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\Lambda_{g_j} = \frac{\partial}{\partial x_j}\Lambda_f$.
(3) For $j = 1,2,3$ set $h_j(\boldsymbol{x}) = \frac{3x_j^2 - \|\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^5}$. Show that $h_j \notin L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and

$$\frac{\partial^2}{\partial x_1^2}\Lambda_f(\varphi) = \lim_{\varepsilon\to 0+}\Big(\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} h_1(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$
$$-\frac{1}{\varepsilon^3}\int_{B_{\mathbb{R}^2}(0,\varepsilon)} (\varphi(-\sqrt{\varepsilon^2-u^2-v^2},u,v) + \varphi(\sqrt{\varepsilon^2-u^2-v^2},u,v))\sqrt{\varepsilon^2-u^2-v^2}\,\mathrm{d}u\,\mathrm{d}v\Big)$$

(4) Show that f „solves the equation $\Delta f = -4\pi\delta_{(0,0,0)}$ in distributions," i.e., $\Delta\Lambda_f = -4\pi\Lambda_{\delta_{(0,0,0)}}$, where $\Delta\Lambda = D^{(2,0,0)}\Lambda + D^{(0,2,0)}\Lambda + D^{(0,0,2)}\Lambda$.

Hint: *(1) On $U(0,1)$ use spherical coordinates. (2) To prove local integrability use again spherical coordinates. Then use Fubini theorem and integration by parts. (3) To disprove local integrability use spherical coordinates. To prove the formula observe that $\frac{\partial^2}{\partial x_1^2}\Lambda_f(\varphi) = \frac{\partial}{\partial x_1}\Lambda_{g_1}(\varphi) = -\Lambda_{g_1}(\frac{\partial \varphi}{\partial x_1}) = -\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^3\setminus B(0,\varepsilon)} g_1 \cdot \frac{\partial\varphi}{\partial x_1}$ and then use Fubini theorem and integration by parts. (4) Take the sum of the formula in (3) and its analogues for $j = 2,3$. Observe that $h_1 + h_2 + h_3 = 0$ outside the origin and that the sum of the second parts may be expressed as $\lim_{\varepsilon\to 0+}\int_{B(0,\varepsilon)} \frac{\partial}{\partial x_1}(x_1\varphi) + \frac{\partial}{\partial x_2}(x_2\varphi)\frac{\partial}{\partial x_3}(x_3\varphi)$.*

Problem 31. Consider the function on $\mathbb{R}^2$ defined by the formula

$$f(t,x) = \begin{cases} \frac{1}{\sqrt{4\pi t}}\exp(-\frac{x^2}{4t}) & t > 0, \\ 0 & t \le 0. \end{cases}$$

(1) Show that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.
(2) Show that $\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = 0$ on the half-plane given by $t > 0$.
(3) Show that $\int_{\mathbb{R}} f(t,x)\,\mathrm{d}x = 1$ for each $t > 0$.
(4) Show that $D^{(1,0)}\Lambda_f - D^{(0,2)}\Lambda_f = \Lambda_{\delta_{(0,0)}}$.

Hint: *(1) follows for example from (3). (3) It is known that $\int_R e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$. (4) Let $\varphi \in \mathscr{D}(\mathbb{R}^2)$. Start by showing that $(D^{(1,0)}\Lambda_f - D^{(0,2)}\Lambda_f)(\varphi) = \lim_{\varepsilon\to 0+}\int_\varepsilon^\infty f(\varepsilon,x)f(\varepsilon,x)\,\mathrm{d}x$. To show that use definitions, Fubini theorem, integration by parts, equality from (2) and Newton-Leibniz formula. Further plug there f, use a suitable substitution and use Lebesgue dominated convergence theorem.*

Problems to Section VII.3 – more on distributions

Problem 32. Let (Λ_n) be a sequence of distributions on Ω such that all of them are of order at most k. Assume that $\Lambda_n \to \Lambda$ in $\mathscr{D}'(\Omega)$. Must Λ be of order at most k as well?

Hint: *Consider the distribution $\Lambda_{1/x}$ on $\mathbb{R}$.*

Problem 33. Determine the supports of distributions from Example 17. Which of them have compact support?

Problem 34. Determine the supports of distributions from Example 18. Which of them have compact support?

Problem 35. Determine the supports of distributions from Example 24. Which of them have compact support?

Problem 36. Determine the supports of distributions from Example 25. Which of them have compact support?

Problems to Section VII.4 – convolution distributions

Problem 37. Let (Λ_n) be a sequence of distributions on Ω such that all of them are of order at most k. Assume that $\Lambda_n \to \Lambda$ in $\mathscr{D}'(\Omega)$. Must Λ be of finite order?

Hint: *Use assertion (d) from Theorem VII.15.*

Problem 38. Let μ be a measure on $\mathbb{R}^d$ (either nonnegative or complex). Show that for each $\varphi \in \mathscr{D}(\mathbb{R}^d)$ we have $\Lambda_\mu * \varphi(\boldsymbol{x}) = \int \varphi(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y})$ for $\boldsymbol{x} \in \mathbb{R}^d$.

Problem 39. Let μ and ν be two complex measures on $\mathbb{R}^d$. Show that the convolution $\Lambda_\mu * \Lambda_\nu$ may be defined and that its result is the distribution $\Lambda_{\mu*\nu}$ where $\mu * \nu$ is the measure on $\mathbb{R}^d$ defined bz the formula

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^d \times \mathbb{R}^d; \boldsymbol{x}+\boldsymbol{y} \in A\}), \quad A \subset \mathbb{R}^d \text{ Borel}.$$

`Hint:` *Show that it fits to the situation (4) from Section VII.4 for $m = n = 0$ (recall that complex measures automatically have bounded variation). For computation use definitions, Fubini theorem an integration with respect to the image of a measure.*

Problem 40. Consider the following distributions on $\mathbb{R}$:

$$U(\varphi) = \sum_{n=1}^{\infty} D^n\varphi(n),\ V(\varphi) = \sum_{n=1}^{\infty} D^n\varphi(n-6),\ W(\varphi) = \sum_{n=1}^{\infty} D^n\varphi(2n), \quad \varphi \in \mathscr{D}(\mathbb{R}).$$

Show that it is possible to define their convolutions

$$U * U, U * V, U * W, V * V, V * W, W * W$$

and compute them.

`Hint:` *Show that it fits to the situation (3) from Section VII.4.*

Problem 41. (1) Show that the convolution $\Lambda_{\chi_{(0,\infty)}} * ((\Lambda_{\delta_0})' * \Lambda_1)$ is well defined and compute it.
(2) Show that the convolution $(\Lambda_{\chi_{(0,\infty)}} * (\Lambda_{\delta_0})') * \Lambda_1$ is well defined and compute it.
(3) Do the two results coincide? What does it say on the associativity of convolution of distributions?

`Hint:` *Use among others Proposition VII.16(g) and Problem 7.*

Problem 42. Let U, V be distributions on $\mathbb{R}^d$ with compact support and $\varphi \in \mathscr{D}(\mathbb{R}^d)$. Show that $(U * V) * \varphi = U * (V * \varphi)$.

Problem 43. Let U, V, W be three distributions on $\mathbb{R}^d$ such that at least two of them have compact support. Show that $(U * V) * W = U * (V * W)$.

Problems to Section VII.5 – Tempered distributions

Problem 44. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be a nonnegative function, such that the distribution Λ_f is tempered. Show that there exist $C > 0$ and $N \in \mathbb{N}_0$ such that

$$\forall R \geq 1\colon \int_{-R}^{R} f \leq C(1+R)^N.$$

`Hint:` *Choose a nonnegative $\psi \in \mathscr{D}((-2,2))$, which equals 1 on $[-1,1]$. Use Proposition VII.18(b) and apply the relevant inequality to the function $\psi_R(x) = \psi(\frac{x}{R})$.*

Problem 45. Let μ be a nonnegative measure on $\mathbb{R}$, such that the distribution Λ_μ is tempered. Show that there exist $C > 0$ and $N \in \mathbb{N}_0$ such that

$$\forall R \geq 1\colon \mu([-R,R]) \leq C(1+R)^N.$$

`Hint:` *Proceed in the same way as in preceding problem.*

Problem 46. Which of the following distributions are tempered?
(1) Λ_f, where $f(x) = e^x$, $x \in \mathbb{R}$;
(2) Λ_g, where $g(x) = e^x \cos(e^x)$, $x \in \mathbb{R}$;
(3) $U(\varphi) = \sum_{n=1}^{\infty} n^2\varphi(n)$;
(4) $U(\varphi) = \sum_{n=1}^{\infty} e^n\varphi(n)$;

`Hint:` *(1) Use Problem 44. (2) Note tat g has a bounded antiderivative. (4) Use Problem 45.*

Problem 47. Let $U \in \mathscr{D}'(\mathbb{R}^d)$ be tempered. Show that U is of finite order.

`Hint:` *Use Proposition VII.18(b).*

Problems to Section VII.5 – Fourier transform

Problem 48. Let μ be a (signed or complex) measure on $\mathbb{R}^d$. Show that $\widehat{\Lambda_\mu} = \Lambda_f$ for a bounded continuous function f on $\mathbb{R}^d$ and compute f.

`Hint:` *Use the definition and Fubini theorem. To prove continuity of f use the theorem on continuity with respect to a parameter.*

Problem 49. Compute $\widehat{\Lambda_{\delta_0}}$ and, more generally, $\widehat{\Lambda_{\delta_a}}$ for $a \in \mathbb{R}$.

`Hint:` *Use the preceding problem.*

Problem 50. Compute $\widehat{\Lambda_{\cos}}$ and $\widehat{\Lambda_{\sin}}$.

`Hint:` *Express* cos *and* sin *using the exponential function, use the result of the preceding problem and Theorem VII.25(a).*