

# FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2023/2024

PROBLEMS TO CHAPTER VI

## PROBLEMS TO SECTION VI.1 – GENERAL WEAK TOPOLOGIES

**Problem 1.** Let  $X = C([0, 1])$  be equipped with the topology of pointwise convergence on  $[0, 1]$ . Describe all the continuous linear functionals on  $X$  (i.e., describe  $X^*$ ).

*Hint: Use Theorem VI.4.*

**Problem 2.** Let  $X$  be a normed linear space which is not complete and let  $Y$  be its completion.

- (1) Show that  $X^* = Y^*$  and explain what this equality means.
- (2) Show that the topologies  $\sigma(Y^*, Y)$  and  $\sigma(Y^*, X)$  are different.

*Hint: (2) Use Theorem VI.4.*

**Problem 3.** Let  $X$  be a normed linear space,  $X^*$  its dual and  $X^{**}$  the second dual. Show that the weak and weak\* topologies on  $X^*$  (i.e., the topologies  $\sigma(X^*, X^{**})$  and  $\sigma(X^*, X)$ ) coincide if and only if  $X$  is a reflexive Banach space.

*Hint: Use Theorem VI.4.*

**Problem 4.** Let  $X$  be a normed linear space. Show that the canonical embedding  $\varkappa : X \rightarrow X^{**}$  is a homeomorphism of  $(X, w)$  into  $(X^{**}, w^*)$ .

*Hint: Use Proposition VI.1(6).*

**Problem 5.** By Problem V.39 and the Introduction to functional analysis we know that  $(\ell^p)^* = \ell^\infty$  for any  $p \in (0, 1]$ .

- (1) Show that the topologies  $\sigma(\ell^\infty, \ell^p)$ ,  $p \in (0, 1]$ , are pairwise distinct.
- (2) Let  $0 < p < q \leq 1$ . Which one of the topologies  $\sigma(\ell^\infty, \ell^p)$  and  $\sigma(\ell^\infty, \ell^q)$  is weaker?

*Hint: (1) Use Theorem VI.4. (2) Use Proposition VI.1(6).*

**Problem 6.** Let  $X$  be a vector space and let  $M \subset X^\#$  separate points of  $X$ . Show that the topology  $\sigma(X, M)$  is metrizable if and only if  $M$  does not contain an uncountable linearly independent subset.

*Hint: Use Theorem V.22 and Lemma VI.3.*

## PROBLEMS TO SECTION VI.2 – WEAK TOPOLOGIES ON LOCALLY CONVEX SPACES

**Problem 7.** Let  $X$  be a normed linear space. Show that  $(X, \|\cdot\|)$  is separable if and only if  $(X, w)$  is separable.

*Hint: Use Mazur theorem.*

**Problem 8.** Find an example of a Banach space  $X$  and a convex norm-closed subset of  $X^*$  which is not weak\* closed.

*Hint:* Consider for example  $X = c_0$ , hence  $X^* = \ell^1$ , and the closed convex hull of canonical unit vectors in  $\ell^1$ . Another examples follow by Goldstine theorem.

**Problem 9.** Let  $X$  and  $Y$  be LCS and let  $T : X \rightarrow Y$  be a continuous linear mapping. Show that  $T$  is continuous as a mapping of  $(X, w)$  to  $(Y, w)$  as well.

*Hint:* Use Proposition VI.1(6).

**Problem 10.** Let  $X$  and  $Y$  be LCS and let  $T : X \rightarrow Y$  be a continuous linear mapping. For  $\varphi \in Y^*$  define a mapping  $T'\varphi : X \rightarrow \mathbb{F}$  by  $T'\varphi = \varphi \circ T$ .

- (1) Show that  $T'\varphi \in X^*$  for each  $\varphi \in Y^*$ .
- (2) Show that the mapping  $T' : \varphi \mapsto T'\varphi$  is a linear mapping of  $Y^*$  into  $X^*$ .
- (3) Show that the mapping  $T'$  is continuous from  $(Y^*, w^*)$  to  $(X^*, w^*)$ .

*Hint:* (3) Use Proposition VI.1(6).

**Problem 11.** Let  $X$  and  $Y$  be normed linear spaces and let  $T : Y^* \rightarrow X^*$  be a bounded linear operator. Show that there exists  $S \in L(X, Y)$  such that  $T = S'$ , if and only if  $T$  is continuous as a mapping of  $(Y^*, w^*)$  into  $(X^*, w^*)$ .

*Hint:* Consider the operator  $T' : X^{**} \rightarrow Y^{**}$  and show that  $T'(\mathcal{K}(X)) \subset \mathcal{K}(Y)$  using Corollary VI.5(c).

**Problem 12.** Let  $X$  be a Hilbert space and let  $(e_n)$  be an orthonormal sequence in  $X$ . Show that the sequence  $(e_n)$  converges weakly to zero.

*Hint:* Use the representation of the dual to a Hilbert space and the Bessel inequality.

**Problem 13.** Let  $X$  be a Hilbert space and let  $(e_\gamma)_{\gamma \in \Gamma}$  be an orthonormal system in  $X$ . Show that the set  $\{e_\gamma; \gamma \in \Gamma\} \cup \{\mathbf{o}\}$  is weakly compact.

*Hint:* Using the representation of the dual to a Hilbert space and the Bessel inequality show that any weak neighborhood of zero contains all the elements of the orthonormal system except for finitely many.

**Problem 14.** Let  $X = c_0(\Gamma)$  or  $X = \ell^p(\Gamma)$ , where  $p \in (1, \infty)$  and  $\Gamma$  is a set. Show that the set  $\{\mathbf{o}\} \cup \{e_\gamma; \gamma \in \Gamma\}$  is weakly compact ( $e_\gamma$  denotes the respective canonical unit vector).

*Hint:* Using the representation of  $X^*$  show that any weak neighborhood of zero contains all the canonical unit vectors except for finitely many.

**Problem 15.** Let  $X = \ell^1(\Gamma)$ , where  $\Gamma$  is a set. Show that the set  $\{e_\gamma; \gamma \in \Gamma\}$  is closed and discrete in the weak topology (i.e., each of its subsets is weakly closed in  $X$ ).

*Hint:* Using the representation  $X^* = \ell^\infty(\Gamma)$  show that each point of this set is isolated (in the weak topology) and that its complement is weakly open.

**Problem 16.** Let  $X = C([0, 1])$ . Consider three topologies of  $X$  – the norm topology (i.e., the topology generated by the supremum norm), the weak one (i.e., the weak topology of the space  $(X, \|\cdot\|_\infty)$  – let us denote it by  $w$ ) and the topology of pointwise convergence on  $[0, 1]$  (denote it by  $\tau_p$ ).

- (1) Find a sequence  $(f_n)$  in  $X$  converging to zero in  $\tau_p$ , which is not bounded in the norm.
- (2) Show that there exists a  $\tau_p$ -bounded set which is not norm-bounded.
- (3) Let  $(f_n)$  be a norm-bounded sequence in  $X$  and let  $f \in X$ . Show that  $f_n \xrightarrow{w} f$  if and only if  $f_n \xrightarrow{\tau_p} f$ .
- (4) Does the equivalence in (3) hold without the assumption of norm boundedness?

*Hint:* (2) Use the sequence from (1). (3) Use Riesz theorem on the representation of  $C([0, 1])^*$  and Lebesgue dominated convergence theorem. (4) Consider the sequence from (1).

**Problem 17.** Show that in the space  $\ell^1$  weak and norm convergences of sequences coincide (i.e.,  $\ell^1$  enjoys the **Schur property**).

*Hint:* Proceed by contradiction: If not, then in  $\ell^1$  there exists a sequence  $(\mathbf{x}_k)$  weakly converging to zero and a number  $c > 0$  such that  $\|\mathbf{x}_k\| > c$  for each  $k \in \mathbb{N}$ . Since  $(\mathbf{x}_k)$  is bounded, without loss of generality  $\|\mathbf{x}_k\| = 1$  for each  $k$ . Weak convergence implies the convergence on each coordinate. By induction construct increasing sequences of natural numbers  $(k_j)$  and  $(m_j)$  such that  $\sum_{l=m_j+1}^{m_{j+1}} |x_{k_j}(l)| > \frac{3}{4}$ . Further find  $\varphi \in \ell^\infty = (\ell^1)^*$  such that  $|\varphi(x_{k_j})| > \frac{1}{2}$  for each  $j$  and deduce a contradiction.

**Problem 18.** Show that the spaces  $c_0$ ,  $\ell^p$  for  $p \in (1, \infty]$  and  $\mathcal{C}([0, 1])$  fail the Schur property.

*Hint:* In any of these spaces find a sequence on the unit sphere weakly converging to zero. For  $\mathcal{C}([0, 1])$  use the description from Problem 16(3).

**Problem 19.** Show that an infinite-dimensional Hilbert space fails the Schur property.

*Hint:* Use Problem 12.

**Problem 20.** Show that the space  $L^1([0, 1])$  fails the Schur property.

*Hint:* Let  $T : L^2([0, 1]) \rightarrow L^1([0, 1])$  be the identity. Consider the ON basis  $(f_n)$  of the space  $L^2([0, 1])$  known from the theory of Fourier series and consider the sequence  $(Tf_n)$ .

**Problem 21.** Let  $X$  be normed linear space of infinite dimension.

- (1) Show that any weak neighborhood of zero contains a nontrivial vector subspace of  $X$ .
- (2) Show that  $S_X$  is a weakly dense subset of  $B_X$ .

*Hint:* (1) Show that any weak neighborhood of zero contains the intersection of kernels of a finite number of functionals, and that this is a nontrivial vector subspace. (2) Use (1).

**Problem 22.** Let  $X$  be normed linear space of infinite dimension.

- (1) Show that any weak\* neighborhood of zero in  $X^*$  contains a nontrivial vector subspace of  $X^*$ .
- (2) Show that  $S_{X^*}$  is weak\* dense subset of  $B_{X^*}$ .

*Hint:*  $X^*$  has infinite dimension as well and the weak\* topology is weaker than the weak one, hence one can apply Problem 21.

**Problem 23.** Let  $X$  be a normed linear space. Show that the following assertions are equivalent:

- (i)  $\dim X < \infty$ .
- (ii) The weak and norm topologies on  $X$  coincide.
- (iii) The weak\* and norm topologies on  $X^*$  coincide.

PROBLEMS TO SECTION VI.3 – POLARS AND THEIR APPLICATIONS

**Problem 24.** Let  $X$  be a separable normed linear space. Show that  $(X^{**}, w^*)$  is separable.

*Hint: Use Goldstine theorem.*

**Problem 25.** Show that  $((\ell^\infty)^*, w^*)$  is separable.

*Hint:  $\ell^\infty = (\ell^1)^*$ .*

**Problem 26.** Let  $X$  be a metrizable LCS. Show that  $(X^*, w^*)$  is  $\sigma$ -compact (i.e., it is the union of countably many compact subsets).

*Hint: Use Theorem VI.14 and a countable base of neighborhoods of zero in  $X$ .*

**Problem 27.** Let  $X$  be a non-complete normed linear space and let  $Y$  be its completion. By Problem 2 we know that  $X^* = Y^*$  and  $\sigma(Y^*, X) \neq \sigma(Y^*, Y)$ . Show that on the unit ball  $B_{Y^*}$  the topologies  $\sigma(Y^*, X)$  and  $\sigma(Y^*, Y)$  coincide.

*Hint: By Corollary VI.16 we know that  $(B_{Y^*}, \sigma(Y^*, Y))$  is compact and the topology  $\sigma(Y^*, X)$  is a weaker Hausdorff topology.*

**Problem 28.** Consider the space  $\ell^\infty$  as the dual to  $\ell^1$ . Show that on the unit ball of  $\ell^\infty$  the weak\* topology  $\sigma(\ell^\infty, \ell^1)$  coincides with the topology of pointwise convergence (i.e. with the topology generated by the seminorms  $\mathbf{x} = (x_k)_{k=1}^\infty \mapsto |x_n|$ ,  $n \in \mathbb{N}$ ).

*Hint: Use Problem 27.*

**Problem 29.** Consider the space  $\ell^1$  as the dual to  $c_0$ . Show that on the unit ball of  $\ell^1$  the weak\* topology  $\sigma(\ell^1, c_0)$  coincides with the topology of pointwise convergence.

*Hint: Use Problem 27.*

**Problem 30.** Let  $p \in (1, \infty)$ . Show that on the unit ball of  $\ell^p$  the weak topology coincides with the topology of pointwise convergence.

*Hint: Use Problem 27 and the reflexivity of  $\ell^p$ .*

**Problem 31.** Show that on the unit ball of  $c_0$  the weak topology coincides with the topology of pointwise convergence.

*Hint: Use Problems 4 and 28.*

**Problem 32.** Let  $X$  be a LCS and let  $X^*$  be its dual. For a nonempty  $A \subset X^*$  define

$$q_A(x) = \sup\{|f(x)|; f \in A\}, \quad x \in X.$$

- (1) Show that  $A$  is  $\sigma(X^*, X)$ -bounded if and only if  $q_A(x) < \infty$  for each  $x \in X$ .
- (2) Let  $A$  be  $\sigma(X^*, X)$ -bounded. Show that  $q_A$  is a seminorm on  $X$ .
- (3) Must  $q_A$  be continuous on  $X$ ?
- (4) Let  $U$  be an absolutely convex neighborhood of zero in  $X$ . Show that  $p_U = q_U \circ$  (where  $p_U$  is the Minkowski functional).

*Hint: (3) Take an infinite-dimensional Banach space  $X$  equipped with the weak topology and  $A = B_{X^*}$ . (4) Use the bipolar theorem.*

**Problem 33.** Let  $X$  be a normed linear space,  $C > 0$  and  $f, g \in S_{X^*}$ . Let  $\|f|_{\ker g}\| \leq C$ . Show that there exists  $\alpha \in \mathbb{F}$ ,  $|\alpha| = 1$  such that  $\|f - \alpha g\| \leq 2C$ .

*Hint:* If  $C \geq 1$  the statement is trivial, so suppose  $C < 1$ . By the Hahn-Banach theorem there exists  $\tilde{f} \in X^*$ , such that  $\|\tilde{f}\| \leq C$  and  $\tilde{f} = f$  on  $\ker g$ . Since  $\ker g \subset \ker(f - \tilde{f})$ , there is  $\beta \in \mathbb{F}$  such that  $f - \tilde{f} = \beta g$ . Show that one can take  $\alpha = \frac{\beta}{|\beta|}$ .

**Problem 34.** Let  $X$  be a Banach space. Let  $f : X^* \rightarrow \mathbb{F}$  be a linear functional such that  $f|_{B_{X^*}}$  is a weak\* continuous mapping. Show that  $f \in \mathcal{K}(X)$ .

*Hint:* Since  $f(B_{X^*})$  is a compact subset of  $\mathbb{F}$ , one gets  $f \in X^{**}$ . The case  $f = 0$  is trivial, so without loss of generality  $\|f\| = 1$ . For  $\varepsilon \in (0, 1)$  set  $A_\varepsilon = \{x^* \in B_{X^*}; \operatorname{Re} f(x^*) \geq \varepsilon\}$  and  $B_\varepsilon = \{x^* \in B_{X^*}; \operatorname{Re} f(x^*) \leq -\varepsilon\}$ . Then  $A_\varepsilon$  and  $B_\varepsilon$  are nonempty disjoint weak\* compact convex sets, hence by the separation theorem there exists  $g \in \mathcal{K}(X)$  such that  $\sup \operatorname{Re} g(B_\varepsilon) < \inf \operatorname{Re} g(A_\varepsilon)$ . Deduce that  $\|f|_{\ker g}\| \leq \varepsilon$ . Using Problem 33 then show that  $f$  belongs to the norm closure of  $\mathcal{K}(X)$ , so to  $\mathcal{K}(X)$ .

**Problem 35.** Is the statement of the previous problem valid also for non-complete spaces?

*Hint:* Use Problem 27.