FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2023/2024

PROBLEMS TO CHAPTER V

Problems to Section V.1 – locally convex topologies and their generation

Problem 1. Let X be a vector space. Let $\mathcal{U}$ be the family of all the absolutely convex absorbing subsets of X.

(1) Show that $\mathcal{U}$ is a base of neighborhoods of zero in a Hausdorff locally convex topology $\mathcal{T}$ on X.
(2) Show that this topology $\mathcal{T}$ is the strongest locally convex topology on X.
(3) Show that any convergent sequence in $(X, \mathcal{T})$ is contained in a finite-dimensional subspace.
(4) Show that $\mathcal{T}$ is generated by the family of all seminorms on X.

`Hint:` *(3) Suppose it is not the case. Then there exists a linearly independent sequence (x_n) which converges to zero. Complete this sequence to an algebraic basis of X. Describe an absolutely convex absorbing set, not containing any of the vectors x_n.*

Problem 2. (1) Show that the convex hull of a balanced subset of a vector space is again balanced, and hence absolutely convex.
(2) Show that the balanced hull of a convex set need not be convex.

`Hint:` *(2) Consider a suitable segment in $\mathbb{R}^2$.*

Problem 3. Let X be a LCS and let $A \subset X$ be a balanced set with nonempty interior.

(1) Show that $\operatorname{int} A$ is balanced if and only if $0 \in \operatorname{int} A$.
(2) Show on a counterexample that $\operatorname{int} A$ need not be balanced.

Problem 4. Let $(X, \mathcal{T})$ be a LCS and let $A \subset X$ be nonempty. Show that

$$\overline{A} = \bigcap \{A + U; U \in \mathcal{T}(0)\}.$$

Problem 5. Let $(X, \mathcal{T})$ be a non-Hausdorff LCS.

(1) Denote $Z = \overline{\{\boldsymbol{o}\}} = \bigcap \mathcal{T}(0)$. Show that Z is a vector subspace of X.
(2) Let $Y = X/Z$ be the quotient vector space and let $q : X \to Y$ be the canonical quotient mapping. Let $\mathcal{R}$ be the quotient topology on Y (i.e., $\mathcal{R} = \{U \subset Y; q^{-1}(U) \in \mathcal{T}\}$). Show that $(Y, \mathcal{R})$ is a HLCS.

Problems to Section V.2 – bounded sets, continuous linear mappings

Problem 6. Let X be a LCS and let $A \subset X$. Show that A is bounded if and only if each countable subset of A is bounded.

Problem 7. Let X be a LVS and let $A, B \subset X$ be bounded sets. Show that the sets $A \cup B$, $A + B$, $\overline{A}$, $\mathrm{b}(A)$, $\operatorname{co} A$ and $\operatorname{aco} A$ are bounded as well.

Problem 8. Let X be a normed linear space and $A \subset X$. Show that A is bounded as a subspace of the LCS X if and only if it is bounded in the metric generated by the norm.

Problem 9. Let X be a LCS whose topology is generated by a translation invariant metric ρ.

(1) Show that any set $A \subset X$ bounded in X is bounded in the metric ρ as well.
(2) Show that a set $A \subset X$ bounded in the metric ρ need not be bounded in the TVS X.

`Hint:` *(2) The metric ρ itself may be bounded.*

Problem 10. Consider the space $(X, \mathcal{T})$ from Problem 1. Show that any linear functional $L : X \to \mathbb{F}$ is continuous.

Problem 11. Let $X = \mathbb{F}^\Gamma$ and let $A \subset X$. Show that A is bounded in X if and only if it is „pointwise bounded“, i.e., if and only if the set $\{x(\gamma); x \in A\}$ is bounded in $\mathbb{F}$ for each $\gamma \in \Gamma$.

Problem 12. Let X be a LCS and let (x_n) be a sequence of elements of X. Show that the sequence (x_n) is bounded in X if and only if for any sequence (λ_n) in $\mathbb{F}$ one has $\lambda_n \to 0 \Rightarrow \lambda_n x_n \to \boldsymbol{o}$.

Problem 13. Let X be a metrizable LCS and let (x_n) be a sequence of elements of X. Show that there exists a sequence of strictly positive numbers (λ_n) such that $\lambda_n x_n \to \boldsymbol{o}$.

`Hint:` *Let ρ be a metric generating the topology on X. Show and then use that $\lim_{t\to 0+} \rho(\boldsymbol{o}, tx) = 0$ for each $x \in X$.*

Problem 14. Is the assertion from Problem 13 true also for non-metrizable LCS?

`Hint:` *Consider the space from Problem 1.*

Problem 15. Let X be a LCS, whose topology is generated by a translation invariant metric ρ. Let (x_n) be a sequence of elements X converging to zero. Show that there exists a sequence of positive numbers (λ_n) such that $\lambda_n \to \infty$ and $\lambda_n x_n \to \boldsymbol{o}$.

`Hint:` *By the translation invariance of ρ it follows $\rho(\boldsymbol{o}, nx) \le n\rho(\boldsymbol{o}, x)$ for $x \in X$ and $n \in \mathbb{N}$.*

Problem 16. Is the assertion from the previous problem valid for a general LCS?

`Hint:` *Consider, e.g., $X = c_0$ or $X = \ell^p$ for $p \in (1, \infty)$ with the weak topology (see Section VI.1), let (x_n) be the sequence of canonical unit vectors.*

Problems to Section V.3 – finite-dimensional and infinite-dimensional spaces

Problem 17. Let X be a metrizable LCS of infinite dimension. Show that there exists a discontinuous linear functional on X.

`Hint:` *Use an algebraic basis of X and Problem 13.*

Problem 18. Is there a a discontinuous linear functional on each infinite-dimensional HLCS?

`Hint:` *Use Problem 10.*

Problems to Section V.4 – metrizability of TVS

Problem 19. Show that the space $\mathbb{F}^\Gamma$ is metrizable if and only if Γ is countable.

`Hint:` *To prove the 'if part', use Proposition V.21. To prove the 'only if part' assume that Γ is uncountable and show that there does not exist a countable base of neighborhoods of zero. To this end use the definition of product topology, in particular the fact that a basic neighborhood of zero is defined using a finite number of coordinates.*

Problem 20. Show that the space $\mathbb{F}^\Gamma$ is normable if and only if Γ is finite.

`Hint:` *To prove the 'only if part' assume that Γ is infinite and prove that no neigborhood of zero is bounded. To this end use the definition of product topology, in particular the fact that a basic neighborhood of zero is defined using a finite number of coordinates.*

Problem 21. Show that the space $\mathcal{C}(\mathbb{R},\mathbb{F})$ from Example V.1(3) is not normable.

`Hint:` *Prove that no neighborhood of zero is bounded. To this end use the seminorms from Example V.6(3).*

Problem 22. Show that the space $H(\Omega)$ from Example V.1(4) is not normable.

`Hint:` *Proceed similarly as in the previous problem.*

Problem 23. Consider the space $\mathcal{C}(T,\mathbb{F})$ of continuous functions on a Tychonoff space T equipped with the topology of uniform convergence on compact subsets of T (see Example V.6(4)).

(1) Show that the space $\mathcal{C}(T,\mathbb{F})$ is metrizable if and only if there is a sequence (K_n) of compact subsets of T such that for any compact set $K \subset T$ there is $n \in \mathbb{N}$ with $K \subset K_n$.
(2) Assume that T is not σ-compact. Show that $\mathcal{C}(T,\mathbb{F})$ is not metrizable.
(3) Assume that T is locally compact. Show that $\mathcal{C}(T,\mathbb{F})$ is metrizable if and only if T is σ-compact.
(4) Let $T = \mathbb{Q}$ (with the topology inherited from $\mathbb{R}$). Show that T is σ-compact but $\mathcal{C}(T,\mathbb{F})$ is not metrizable.

`Hint:` *(1) To prove the 'if part', use Proposition V.21. To prove the 'only if part' assume that such a sequence (K_n) does not exist and show that there does not exist a countable base of neighborhoods of zero. This may be shown by contradiction. Assume (U_n) is a base of neighborhoods of zero, each U_n may be defined using a compact set $K_n \subset T$. Then there is $K \subset T$ compact such that $K \setminus K_n \neq \emptyset$ for $n \in \mathbb{N}$. Let $V = \{f; \|f|_K\|_\infty < 1\}$. Using complete regularity show that no U_n is contained in V. (3) Assume that T is locally compact and σ-compact. Show that there is a sequence (K_n) of compact subsets of T such that $T = \bigcup_n K_n$ and $K_n \subset \operatorname{int} K_{n+1}$ for each $n \in \mathbb{N}$ and that this sequence satisfies the condition from (1). (4) Let (K_n) be a sequence of compact subsets of $\mathbb{Q}$. Using the fact that any compact subset of $\mathbb{Q}$ is nowhere dense construct a sequence (x_n) in $\mathbb{Q}$ such that $x_n \in \mathbb{Q} \setminus K_n$ and (x_n) converges to 0.*

Problem 24. Show that the space $\mathcal{C}(T,\mathbb{F})$ with the topology of uniform convergence on compact subset of a Tychonoff space T (see Example V.6(4)) is normable if and only if T is compact.

`Hint:` *Assume T is not compact. Let U be a neighborhood of zero defined using a compact set $K \subset T$. Let $x \in T \setminus K$. Show that $f \mapsto |f(x)|$ is a continuous seminorm which is not bounded on U.*

Problems to Section V.5 - Fréchet spaces, totally bounded sets

Problem 25. Let $(X, \|\cdot\|)$ be a Banach space. Let $(|||\cdot|||_k)$ be a sequence of functions $X \to [0,+\infty]$ satisfying the following properties:

- $|||\boldsymbol{o}|||_k = 0$;
- $\forall x \in X\, \forall \alpha \in \mathbb{F} \setminus \{0\} \colon |||\alpha x|||_k = |\alpha| \cdot |||x|||_k$;
- $\forall x, y \in X \colon |||x+y|||_k \le |||x|||_k + |||y|||_k$;
- the function $|||\cdot|||_k$ is lower semicontinuous, i.e. the set $\{x \in X; |||x|||_k \le c\}$ is closed for each $c \in \mathbb{R}$;
- $\forall x \in X : \|x\| \le |||x|||_1 \le |||x|||_2 \le |||x|||_3 \le \dots$

Set $Y = \{x \in X; \forall k \in \mathbb{N} \colon |||x|||_k < +\infty\}$.

(1) Show that Y is a linear subspace of X.
(2) Show that Y is a Fréchet space if it is equipped with the locally convex topology generated by the sequence of norms $(|||\cdot|||_k)$.

`Hint:` *(2) Use Proposition V.21 and the method of proof of Proposition I.5 to each of the norms* $|||\cdot|||_k$.

Problem 26. Let $(X_n, \|\cdot\|_n)$ be a sequence of Banach spaces and for each $n \in \mathbb{N}$ a quotient mapping $P_n : X_{n+1} \to X_n$ is given Set

$$Y = \{(x_n); \forall n \in \mathbb{N} \colon (x_n \in X_n \;\&\; x_n = P_n(x_{n+1}))\}.$$

Show that Y is a Fréchet space if equipped by the sequence of seminorms (p_k) defined by

$$p_k((x_n)) = \|x_k\|_k\,, \quad (x_n) \in Y.$$

`Hint:` *Use Proposition V.21.*

Problem 27. Using Problem 26 show that the spaces $C(\mathbb{R}, \mathbb{F})$ and $H(\Omega)$ (se Example V.1(3,4)) are Fréchet spaces.

Problem 28. Let X be a LCS and let $A, B \subset X$ be totally bounded subsets. Show that the sets $A \cup B$, $A + B$, $\overline{A}$, $\mathrm{b}(A)$ are totally bounded as well.

Problem 29. Let X be a LCS, whose topology is generated by a translation invariant metric ρ. Show that a set $A \subset X$ is totally bounded in the TVS X if and only if it is totally bounded in the metric ρ.

Problems to Section V.6 – extension and separation theorems

Problem 30. Let X be a normed linear space of infinite dimension. Show that in X there exist two disjoint convex sets which are dense in X (and hence they cannot be separated by a nonzero element of X^*).

`Hint:` *Use the existence of a discontinuous linear functional.*

Problem 31. Let $X = \mathcal{C}([0,1])$ be equipped with the L^2-norm (i.e., $\|f\| = \left(\int_0^1 |f|^2\right)^{1/2}$). For $\alpha \in \mathbb{R}$ define $Y_\alpha = \{f \in X; f(0) = \alpha\}$. Show that $(Y_\alpha; \alpha \in \mathbb{R})$ is a pairwise disjoint family if dense convex sets. Show that for $\alpha \ne \beta$ the sets Y_α and Y_β cannot be separated by a nonzero element of X^*.

Problem 32. Let $X = c_0$ or $X = \ell^p$ for some $p \in [1,\infty)$ (consider the real version of these spaces). Let $\boldsymbol{x} = (x_n) \in X$ be an element with all the coordinates strictly positive and let $\boldsymbol{y} = (\frac{x_n}{n}) \in X$. Set

$$A = \{\boldsymbol{z} = (z_n) \in X; \forall n \in \mathbb{N} : z_n \geq 0\}, \qquad B = \{-\boldsymbol{x} + t\boldsymbol{y}; t \in \mathbb{R}\}.$$

Show that A and B are disjoint closed subsets of X, which cannot be separated by a nonzero element of X^*.

`Hint:` *Proceed by contradiction: Let $f \in X^* \setminus \{0\}$ satisfy $\sup f(B) \leq \inf f(A)$. Show that necessarily $f \geq 0$ on A and $\inf f(A) = 0$. The functional f can be represented by an appropriate sequence (by an element of ℓ^1 or ℓ^q where $\frac{1}{p} + \frac{1}{q} = 1$), show that all the entries of this sequence have to be non-negative. By the assumption $\inf f(B) \leq 0$ deduce $f(\boldsymbol{y}) = 0$, hence $f = 0$, a contradiction.*

Further problems – metric vector spaces

A **metric vector spaces** (shortly **MVS**) is a vector space X over $\mathbb{F}$ equipped by a metric ρ, such that the operations on X are continuous with respect to ρ, i.e.:

- $x_n \to x$, $y_n \to y$ in $X \implies x_n + y_n \to x + y$ in X;
- $x_n \to x$ in X, $\lambda_n \to \lambda$ in $\mathbb{F} \implies \lambda_n x_n \to \lambda x$ in X.

Clearly, a MVS is a special case of a TVS. A MVS need not be locally convex, as witnessed for example by the spaces L^p from Example V.1(5).

Problem 33. Let X be the space of all the Lebesgue measurable functions on $[0,1]$ (with values in $\mathbb{F}$; we identify the functions, which are equal almost everywhere). For $f, g \in X$ set

$$\rho(f,g) = \int_0^1 \min\{1, |f-g|\}.$$

(1) Show that ρ is a metric making X a MVS.
(2) Show that the convergence of sequences in the metric ρ coincide with the convergence in measure.
(3) Is the resulting topology locally convex?

`Hint:` *(3) Show that for each $r > 0$ the convex hull of the set $\{f \in X; \rho(f,0) < r\}$ is the whole X.*

Problem 34. Let X be a vector space over $\mathbb{F}$ and let $q : X \to [0,\infty)$ be an F-**norm** on X, i.e., a mapping with the following properties:

- $q(x) = 0 \iff x = 0$;
- $\forall x \in X \forall \lambda \in \mathbb{F}, |\lambda| \leq 1$: $q(\lambda x) \leq q(x)$;
- $\forall x, y \in X$: $q(x+y) \leq q(x) + q(y)$;
- $\forall x \in X$: $\lim\limits_{t \to 0+} q(tx) = 0$.

Show that the formula $\rho(x,y) = q(x-y)$ defines a translation invariant metric on X making X a MVS.

Problem 35. Let X be a vector space over $\mathbb{F}$, let $p \in (0,1)$ and let $q : X \to [0,\infty)$ be a p-**norm** on X, i.e., a mapping with the following properties:

- $q(x) = 0 \iff x = 0$;
- $\forall x \in X \forall \lambda \in \mathbb{F}\colon q(\lambda x) = |\lambda|^p q(x)$;
- $\forall x, y \in X\colon q(x+y) \le q(x) + q(y)$.

Show that the formula $\rho(x,y) = q(x-y)$ defines a translation invariant metric on X making X a MVS.

`Hint:` *Show that q is an F-norm.*

Problem 36. Let $p \in (0,1)$. Show that the function $f \mapsto \|f\|_p = \int |f|^p \, \mathrm{d}\mu$ is a p-norm on $L^p(\mu)$.

Problem 37. Let X be a MVS whose metric ρ is induced by a p-norm for some $p \in (0,1)$. Show that a set $A \subset X$ is bounded in X if and only if it is bounded in the metric ρ.

Problem 38. Let $X = L^p([0,1])$ where $p \in (0,1)$.

(1) Show that $\operatorname{co}\{f \in X; \|f\|_p < r\} = X$ for each $r > 0$.
(2) Show that there exists a bounded subset of X with unbounded convex hull.
(3) Show that the unique continuous linear functional on X is constant zero (i.e., $X^* = \{0\}$).

`Hint:` *(1) Let $f \in X$ and $n \in \mathbb{N}$. Show that there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\int_{t_{j-1}}^{t_j} |f|^p = \frac{1}{n}\int_0^1 |f|^p$ for each j. Next show that for n large enough the p-normu of functions $nf\chi_{(t_{j-1},t_j)}$ is less than r. (2) Use (1) and the previous problem. (3) Use (1).*

Problem 39. Let $X = \ell^p$, where $p \in (0,1)$. Show that for any sequence $\boldsymbol{x} = (x_n) \in \ell^\infty$ the formula

$$\varphi_{\boldsymbol{x}}(\boldsymbol{y}) = \sum_{n=1}^{\infty} x_n y_n, \quad \boldsymbol{y} = (y_n) \in \ell^p,$$

defines a continuous linear fuctional on ℓ^p. Show that the mapping $\boldsymbol{x} \mapsto \varphi_{\boldsymbol{x}}$ is a linear bijection of ℓ^∞ onto X^*.

Problem 40. Let $p \in (0,1)$. Show that ℓ^p is isomorphic (even linearly isometric) to a subspace of $L^p([0,1])$. Using the two previous problems demonstrate on a counterexample that a continous linear functional on a subspace of a MVS need not admit a continous linear extension to the whole space.

Problem 41. Let $X = L^p([0,1])$, where $p \in (0,1)$. Choose strictly positive numbers ε, η and δ such that $p < \frac{1}{1+\varepsilon}$, $\frac{\eta}{\varepsilon} < p$, $\delta < \varepsilon$ and $\frac{\eta}{\varepsilon - \delta} < p$. For $n \in \mathbb{N}$ set $x_n = \frac{1}{n^{1+\eta}}$, $f_n = n^{1+\varepsilon}\chi_{(x_{n+1},x_n)}$ and $t_n = \frac{1}{n^{1+\delta}}$.

(1) Show that the set $K = \{0, f_1, f_2, f_3, \dots\}$ is compact in X.
(2) Show that $\operatorname{co} K$ is unbounded in X.

`Hint:` *(1) Show that $f_n \to 0$ in $L^p([0,1])$. (2) Consider the elements $\frac{t_1 f_1 + \cdots + t_n f_n}{t_1 + \cdots + t_n}$.*