## VI.2 Weak topologies on locally convex spaces

**Remark.** In this section it is essential that we deal with locally convex spaces. For general TVS none of the results is true.

**Theorem 6** (Mazur theorem). *Let $X$ be a LCS and let $A \subset X$ be a convex set. Then:*

(a) $\overline{A}^w = \overline{A}$.

(b) *$A$ is closed if and only if it is weakly closed.*

**Corollary 7.** *Let $X$ be a metrizable LCS and let $(x_n)$ be a sequence in $X$ weakly converging to a point $x \in X$. Then there is a sequence $(y_n)$ in $X$ such that*

- *$y_n \in \operatorname{co}\{x_k; k \geq n\}$ for each $n \in \mathbb{N}$;*
- *$y_n \to x$ in (the original topology of) $X$.*

**Remark:** If $X$ is a LCS, $(x_n)$ a sequence in $X$ and $x \in X$, then

$$x_n \overset{\sigma(X,X^*)}{\longrightarrow} x \iff \forall f \in X^*\colon f(x_n) \longrightarrow f(x).$$

In particular, if $X$ is a normed linear space, pak

$$x_n \overset{\sigma(X,X^*)}{\longrightarrow} x \iff x_n \overset{w}{\longrightarrow} x,$$

where the notation from Section II.4 is used. Thus the **weak convergence** means the *convergence in the weak topology.*

**Theorem 8** (boundedness and weak boundedness). *Let $X$ be a LCS and let $A \subset X$. Then $A$ is bounded in $X$ if and only if it is bounded in $\sigma(X, X^*)$.*

**Remark:** By Lemma V.15 we deduce that $A$ is bounded in $\sigma(X, X^*)$ if and only if each $f \in X^*$ is bounded on $A$.

**Proposition 9** (weak topology on a subspace). *Let $X$ be a LCS and let $Y \subset\subset X$. Then the weak topology $\sigma(Y, Y^*)$ coincides with the restriction of the weak topology $\sigma(X, X^*)$ to $Y$.*

**Remark:** It follows from the remark after Corollary 7 that

- Theorem 6 and Corollary 7 generalize Proposition II.26.
- Theorem 8 generalizes Corollary II.29.
- Proposition 9 generalizes Proposition II.24.