V.6 Extension and separation theorems

Definition. Let X be a LCS over $\mathbb{F}$. By X^* we will denote the vector space of all the continuous linear functionals $f : X \to \mathbb{F}$. The space X^* is called **the dual space** (or **the dual**) of X.

Remarks:

(1) The dual of X is sometimes denoted by X'. The notation used in the literature is not unified. We will use for the 'continuous dual', i.e., for the space of *continuous* linear functionals, the symbol X^*. For the 'algebraic dual', i.e., the space of *all* linear functionals, we will use the symbol $X^{\#}$.

(2) We define X^* to be just a vector space, for the time being we do not equip it with any topology. In the next chapter will consider one natural topology on X^*. Nonetheless, there exist more natural topologies on X^*.

Theorem 31 (Hahn-Banach extension theorem). *Let X be a LCS over $\mathbb{F}$, $Y \subset\subset X$ and $f \in Y^*$. Then there exists $g \in X^*$ such that $g|_Y = f$.*

Corollary 32 (separation from a subspace). *Let X be a LCS, Y a closed subspace of X and $x \in X \setminus Y$. Then there exists $f \in X^*$ such that $f|_Y = 0$ and $f(x) = 1$.*

Corollary 33 (a proof of density using Hahn-Banach theorem). *Let X be a LCS and let $Z \subset\subset Y \subset\subset X$. Then Z is dense in Y if and only if*

$$\forall f \in X^* : f|_Z = 0 \Rightarrow f|_Y = 0.$$

Corollary 34 (the dual separates points). *Let X be a HLCS. Then for any $x \in X \setminus \{0\}$ there exists $f \in X^*$ such that $f(x) \neq 0$.*

Theorem 35 (Hahn-Banach separation theorem). *Let X be a LCS, let $A, B \subset X$ be nonempty disjoint convex subsets.*

(a) *If the interior of A is nonempty, there exist $f \in X^* \setminus \{0\}$ and $c \in \mathbb{R}$ such that*
$$\forall a \in A\, \forall b \in B : \operatorname{Re} f(a) \le c \le \operatorname{Re} f(b).$$

(b) *If A is compact and B is closed, there exist $f \in X^*$ and $c, d \in \mathbb{R}$ such that*
$$\forall a \in A\, \forall b \in B : \operatorname{Re} f(a) \le c < d \le \operatorname{Re} f(b).$$

Corollary 36. *Let X be a LCS, let $A \subset X$ be a nonempty set and let $x \in X$. Then:*

(a) *$x \in X \setminus \overline{\text{co}} A$ if and only if there exists $f \in X^*$ such that*
$$\operatorname{Re} f(x) > \sup\{\operatorname{Re} f(a); a \in A\}.$$

(b) *$x \in X \setminus \overline{\text{aco}} A$ if and only if there exists $f \in X^*$ such that*
$$|f(x)| > \sup\{|f(a)| ; a \in A\}.$$

Remark: The situation for general TVS is the following:

- The dual X^* may be defined in the same way. But it may be trivial – even if X is Hausdoff and nontrivial. If, for example, $X = L^p((0,1))$ for some $p \in (0,1)$, then $X^* = \{0\}$. Therefore Corollary 34 fails for TVS.
- Theorem 31 and Corollaries 32 and 33 fail for TVS.
- Assertion (a) from Theorem 35 holds for TVS as well (with the same proof). Both assertion (b) and Corollary 36 fail for TVS.