V.3 Spaces of finite and infinite dimension

Proposition 17. *Let X be a HLCS of finite dimension.*

(a) *If Y is any LCS and $L : X \to Y$ is any linear mapping, then L is continuous.*
(b) *The space X is isomorphic to $\mathbb{F}^n$, where $n = \dim X$.*

Corollary 18. *Let X be a HLCS. Then any its finite-dimensional subspace is closed.*

Definition. Let $(X, \mathcal{T})$ be a LCS and let $A \subset X$. The set A is said to be **totally bounded** (or **precompact**), if for any $U \in \mathcal{T}(\boldsymbol{o})$ there exists a finite set $F \subset X$ such that $A \subset F + U$.

Remark: Any compact set in any LCS is totally bounded. Any totally bounded set is bounded.

Lemma 19. *Let $(X, \mathcal{T})$ be a LCS and let $A \subset X$. The following assertions are equivalent:*

(1) *A is totally bounded in X.*
(2) *A is totally bounded in (X, p) for any continuous seminorm p on X, i.e.,*

$\forall p$ continuous seminorm on X

$$\forall \varepsilon > 0 \, \exists F \subset X \text{ finite} \, \forall x \in A \, \exists y \in F \colon p(y - x) < \varepsilon.$$

(3) *For any continuous seminorm p on X and any sequence (x_n) in A there is a subsequence (x_{n_k}) which is Cauchy with respect to p, i.e.*

$$\forall \varepsilon > 0 \, \exists k_0 \, \forall k, l \geq k_0 \colon p(x_{n_k} - x_{n_l}) < \varepsilon.$$

(In (2) and (3) it is enough to test the condition for a family of seminorms generating the topology of X.)

Theorem 20. *Let X be a HLCS. The following assertions are equivalent:*

(i) $\dim X < \infty$.
(ii) *There exists a compact neighborhood of zero in X.*
(iii) *There exists a totally bounded neighborhood of zero in X.*

Remark: For general TVS the following statements from this section are valid:

- Proposition 17 and Corollary 18 (no change needed).
- Totally bounded sets are defined in the same way and the following remark holds (no change needed).
- Theorem 20 (no change needed).