V.2 Continuous and bounded linear mappings

Proposition 12. *Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be LCS over $\mathbb{F}$ and let $L: X \to Y$ be a linear mapping. The following assertions are equivalent:*

- (i) *L is continuous.*
- (ii) *L is continuous at $\boldsymbol{o}$.*
- (iii) *L is* **uniformly continuous**, *i.e.,*

$$\forall U \in \mathcal{U}(\boldsymbol{o})\, \exists V \in \mathcal{T}(\boldsymbol{o})\, \forall x, y \in X : x - y \in V \Rightarrow L(x) - L(y) \in U.$$

Proposition 13. *Let X and Y be LCS and let $L: X \to Y$ be a linear mapping. Then L is continuous if and only if*

$$\forall q \text{ a continuous seminorm on } Y\, \exists p \text{ a continuous seminorm on } X\ \ \forall x \in X : q(L(x)) \le p(x).$$

If $\mathcal{P}$ is a family of seminorms generating the topology of X and $\mathcal{Q}$ is a family of seminorms generating the topology of Y, then the continuity of L is equivalent to the condition

$$\forall q \in \mathcal{Q}\, \exists p_1, \dots, p_k \in \mathcal{P}\, \exists c > 0\, \forall x \in X : q(L(x)) \le c \cdot \max\{p_1(x), \dots, p_k(x)\}.$$

Proposition 14. *Let $(X,\mathcal{T})$ be a LCS over $\mathbb{F}$ and let $L: X \to \mathbb{F}$ be a linear mapping. The following assertions are equivalent:*

- (i) *L is continuous.*
- (ii) *$\ker L$ is a closed subspace of X.*
- (iii) *There exists $U \in \mathcal{T}(\boldsymbol{o})$ such that $L(U)$ is a bounded subset of $\mathbb{F}$.*

If $\mathcal{P}$ is a family of seminorms generating the topology of X, the continuity of L is also equivalent to:

- (iv) *$\exists p_1, \dots, p_k \in \mathcal{P}\, \exists c > 0\, \forall x \in X : |L(x)| \le c \cdot \max\{p_1(x), \dots, p_k(x)\}$.*

If L is discontinuous, then $\ker L$ is a dense subspace of X.

Definition. Let $(X,\mathcal{T})$ be a LCS and let $A \subset X$. The set A is said to be **bounded** in $(X,\mathcal{T})$, if for any $U \in \mathcal{T}(\boldsymbol{o})$ there exists $\lambda > 0$ such that $A \subset \lambda U$.

Lemma 15. *Let $(X,\mathcal{T})$ be a LCS and let $A \subset X$. The set A is bounded in X if and only if each continuous seminorm p on X is bounded on A. (It is enough to test it for a family of seminorms generating the topology of X.)*

Proposition 16. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be LCS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. Consider the following two assertions:*

(i) *L is continuous.*
(ii) *For any bounded subset $A \subset X$ its image $L(A)$ is bounded in Y (i.e., L is a* **bounded mapping***).*

Then (i)$\Rightarrow$(ii). In case $\mathcal{T}$ is generated by a translation invariant metric on X, then (i)$\Leftrightarrow$(ii).

Remark. (1) It follows from Theorem 22 in Section V.4 that, whenever a LCS $(X, \mathcal{T})$ is metrizable, i.e., the topology $\mathcal{T}$ is generated by a metric, then this metric can be chosen to be translation invariant.

(2) Equivalence (i)$\Leftrightarrow$(ii) in Proposition 16 fails in general. This follows from a general theorem we arrive later.

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be LCS over $\mathbb{F}$ and let $L : X \to Y$ be a linear mapping. The mapping L is said to be

- an **isomorphism of** X **into** Y if L is continuous, one-to-one and L^{-1} is continuous on $L(X)$;
- an **isomorphism of** X **onto** Y, if L is continuous, one-to-one, onto and L^{-1} is continuous on Y.

The spaces X and Y are said to be **isomorphic** if there is an isomorphism of X onto Y.

Remark: For general TVS the following statements from this section are valid:

- Proposition 12 (no change needed).
- Equivalence of conditions (i)–(iii) from Proposition 14.
- Bounded sets are defined in the same way, Proposition 16 holds (no change needed).
- An obvious analogue of Remark (1) after Proposition 16 holds as well (it follows from an analogue of Theorem 22 with substantially more difficult proof).