IX. Compact convex sets

Convention: In this chapter we consider only vector spaces over $\mathbb{R}$. It causes no harm, as all the definitions and results can be used for complex spaces as well, because only the structure of the real version of the space in question is used.

Definition. Let X be a vector space and let $A \subset X$ be a convex set. A point $x \in A$ is said to be an **extreme point** of A if it is not an interior point of any segment in A, i.e., if

$$\forall a, b \in A \; \forall t \in (0,1) : x = ta + (1-t)b \Rightarrow a = b = x.$$

The set of all extreme points of A is denoted by $\operatorname{ext} A$.

Remark. A point $x \in A$ is an extreme point of a convex set A if and only if it is not the center of any nondegenerate segment in A, i.e., if

$$\forall a, b \in A : x = \frac{1}{2}(a+b) \Rightarrow a = b = x.$$

Examples 1. *Let $X = \mathbb{R}^2$. Then:*

(1) *If $A \subset \mathbb{R}^2$ is a convex polygon, then its extreme points are just its vertices.*
(2) *If $A \subset \mathbb{R}^2$ is a closed disc, then $\operatorname{ext} A$ is its boundary circle.*
(3) *If $A \subset \mathbb{R}^2$ is an open disc, then $\operatorname{ext} A = \emptyset$.*

Definition. Let X be a vector space and let $A \subset X$ be a convex set. A subset $F \subset A$ is said to be a **face** of A if the following two coditions are fulfilled:

- F is a nonempty convex subset of A;
- $\forall a, b \in A : \frac{1}{2}(a+b) \in F \Rightarrow a \in F \; \& \; b \in F$.

Lemma 2 (properties of faces). *Let X be a vector space and let $A \subset X$ be a convex set.*

(a) *$x \in A$ is an extreme point of A if and only if $\{x\}$ is a face of A.*
(b) *If $F_1 \subset A$ is a face of A and $F_2 \subset F_1$ is a face of F_1, then F_2 is a face of A.*
(c) *If, moreover, X is a HLCS and A is a compact set containing at least two points, then there is a closed face $F \subsetneqq A$.*

Theorem 3 (Krein-Milman). *Let X be a HLCS and let $K \subset X$ be a convex compact set. Then*

$$K = \overline{\operatorname{co} \operatorname{ext} K}.$$

In particular, $\operatorname{ext} K \neq \emptyset$ whenever K is nonempty.

Proposition 4 (Minkowski-Carathéodory). *Let X be a HLCS of dimension $n \in \mathbb{N}$ and let $K \subset X$ be a nonempty compact convex set. Then $K = \operatorname{co} \operatorname{ext} K$. Moreover, any point in K can be expressed by a convex combination of at most $n+1$ extreme points of K and these points can be chosen to be affinely independent.*

Example 5. *Let K be a compact Hausdorff space and let $P(K)$ be the set of all Radon probabilities on K considered as a subset of $(\mathcal{C}(K)^*, w^*)$. Then $P(K)$ is a compact convex set and its extreme points are exactly Dirac measures.*

Proposition 6 (Milman). *Let X be a HLCS and $K \subset X$ a convex compact set. If $A \subset K$ is such that $K = \overline{\operatorname{co} A}$, then $\operatorname{ext} K \subset \overline{A}$.*

Proposition 7 (on the barycenter of a measure). *Let X be a HLCS and let $K \subset X$ be a compact convex set.*

(a) *For any $\mu \in P(K)$ there exists a unique $x \in K$ satisfying*

$$\forall f : K \to \mathbb{R} \text{ continuous affine} : f(x) = \int f \, \mathrm{d}\mu.$$

This x is said to be the **barycenter** *of μ and is denoted by $r(\mu)$.*

(b) *The mapping $r : \mu \mapsto r(\mu)$ is a continuous affine mapping of $P(K)$ onto K.*

Theorem 8 (Krein-Milman theorem on integral representation). *Let X be a HLCS and let $K \subset X$ be a compact convex set. Then for each $x \in K$ there exists $\mu \in P(K)$ satisfying $\mu(\overline{\operatorname{ext} K}) = 1$ and $x = r(\mu)$.*

Proposition 9. *Let X be HLCS and let $K \subset X$ be a compact convex set.*

(a) *If K is metrizable, then $\operatorname{ext} K$ is a G_δ subset of K.*

(b) *If $\dim X \leq 2$, then $\operatorname{ext} K$ is a closed subset K.*

Remark. There is a compact convex subset $K \subset \mathbb{R}^3$ such that $\operatorname{ext} K$ is not closed.

Remark. The following Choquet theorem strengthens Theorem 8 in case K is metrizable:

Let X be a HLCS and let $K \subset X$ be a metrizable compact convex set. Then for each $x \in K$ there exists $\mu \in P(K)$ satisfying $\mu(\operatorname{ext} K) = 1$ and $x = r(\mu)$.

There is another version of this theorem for nonmetrizable K, but its formulation is more complicated.