VIII.2 Integrability of vector-valued functions

Definition.

- Let $f : \Omega \to X$ be a simple measurable function of the form $f = \sum_{j=1}^{k} x_j \chi_{E_j}$ (where $E_1, \dots, E_k \in \Sigma$ are pairwise disjoint and $x_1, \dots, x_k \in X$). Let $E \in \Sigma$. We say that f is **integrable over** E, if for each $j \in \{1, \dots, k\}$ one has either $\mu(E \cap E_j) < \infty$ or $x_j = \boldsymbol{o}$. By the **integral of** f **over** E we mean the element of X defined by the formula
$$\int_E f \, \mathrm{d}\mu = \sum_{j=1}^{k} \mu(E \cap E_j) x_j,$$
where by convention $\infty \cdot \boldsymbol{o} = \boldsymbol{o}$. If f is integrable over Ω, it is called **integrable**.
- Let $f : \Omega \to X$ be strongly μ-measurable. The function f is said to be **Bochner integrable** if there exists a sequence (f_n) of simple integrable functions such that
$$\lim_{n \to \infty} \int_\Omega \|f_n(\omega) - f(\omega)\| \, \mathrm{d}\mu(\omega) = 0,$$
where the integral is in the Lebesgue sense. By the **Bochner integral** of f we then mean the element of X defined by
$$(B) \int_\Omega f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu.$$
- A function $f : \Omega \to X$ is said to be **weakly integrable** if $\varphi \circ f$ is integrable (i.e., $\varphi \circ f \in L^1(\mu)$) for each $\varphi \in X^*$.

Proposition 7 (basic properties of the Bochner integral).

(a) *Integrable simple functions form a vector space; and the mapping assigning to a simple integrable function f its integral $\int_\Omega f \, \mathrm{d}\mu$ is linear.*

(b) *Let f be a simple measurable function. Then f is integrable if and only if the function $\omega \mapsto \|f(\omega)\|$ is integrable. In this case*
$$\left\| \int_\Omega f \, \mathrm{d}\mu \right\| \le \int_\Omega \|f(\omega)\| \, \mathrm{d}\mu(\omega).$$

(c) *The limit defining the Bochner integral does exist and does not depend on the choice of the sequence (f_n).*

(d) *Bochner integrable functions form a vector space; and the mapping assigning to a Bochner integrable function its Bochner integral is linear.*

(e) *If $f : \Omega \to X$ is Bochner integrable, then the function $\omega \mapsto \|f(\omega)\|$ is integrable and*
$$\left\| (B) \int_\Omega f \, \mathrm{d}\mu \right\| \le \int_\Omega \|f(\omega)\| \, \mathrm{d}\mu(\omega).$$

(f) *If $f : \Omega \to X$ Bochner integrable, then $\chi_E \cdot f$ is Bochner integrable for each $E \in \Sigma$. (The value $(B) \int_\Omega \chi_E \cdot f \, \mathrm{d}\mu$ is called the* **Bochner integral of** *f* **over** *E and it is denoted by $(B) \int_E f \, \mathrm{d}\mu$.)*

Theorem 8 (a characterization of Bochner integrability). *Let $f : \Omega \to X$ be a strongly μ-measurable function. Then f is Bochner integrable if and only if $\int_\Omega \|f(\omega)\| \, \mathrm{d}\mu(\omega) < \infty$.*

Theorem 9 (Lebesgue dominated convergence theorem for Bochner integral). *Let (f_n) be a sequence of Bochner integrable functions $f_n : \Omega \to X$ almost everywhere converging to a function $f : \Omega \to X$. Let $g : \Omega \to \mathbb{R}$ be an integrable function such that for each $n \in \mathbb{N}$ one has $\|f_n(\omega)\| \le g(\omega)$ for almost all $\omega \in \Omega$. Then f is Bochner integrable and $(B) \int_\Omega f \, \mathrm{d}\mu = \lim_{n \to \infty} (B) \int_\Omega f_n \, \mathrm{d}\mu$.*

Proposition 10 (absolute continuity of Bochner integral). *Let $f : \Omega \to X$ be Bochner integrable. Then:*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall E \in \Sigma : \mu(E) < \delta \Rightarrow \left\| \int_E f \, \mathrm{d}\mu \right\| < \varepsilon.$$

Proposition 11 (weak integral). *Let $f : \Omega \to X$ be weakly integrable. Then the mapping*

$$F(\varphi) = \int_\Omega \varphi \circ f \, \mathrm{d}\mu, \quad \varphi \in X^*,$$

is a continuous linear functional on X^*, i.e., $F \in X^{**}$.

Definition, notation and remarks:

(1) The element $F \in X^{**}$ provided by Proposition 11 is called the **weak integral** (or the **Dunford integral**) of f, it is denoted by $(D) \int_\Omega f \, \mathrm{d}\mu$.
(2) Let $f : \Omega \to X$ be weakly integrable. Then $\chi_E \cdot f$ is weakly integrable for each $E \in \Sigma$. The respective weak integral is denoted by $(D) \int_E f \, \mathrm{d}\mu$.
(3) We say that $f : \Omega \to X$ is **Pettis integrable** if
   - f is weakly integrable and, moreover,
   - the weak integral $(D) \int_E f \, \mathrm{d}\mu$ belongs to $\varkappa(X)$ (where $\varkappa : X \to X^{**}$ is the canonical embedding) for each $E \in \Sigma$.

   The Pettis integral of f over E is then the respective element of X and it is denoted by $(P) \int_E f \, \mathrm{d}\mu$. I.e., for $x \in X$ then one has

$$x = (P) \int_E f \, \mathrm{d}\mu \Leftrightarrow \forall \varphi \in X^* : \varphi(x) = \int_E \varphi \circ f \, \mathrm{d}\mu.$$

Remarks:

(1) In order that f is Pettis integrable, $(D) \int_E f \, \mathrm{d}\mu \in \varkappa(X)$ should hold for each $E \in \Sigma$. It is not enough if it is satisfied in case $E = \Omega$.
(2) A weakly integrable function need not be Pettis integrable.
(3) A Pettis integrable function need not be strongly μ-measurable. For example, the function from Example 6(1) is Pettis integrable, its integral is zero, but it is not essentially separably valued.
(4) Any Bochner integrable function is Pettis integrable (this follows from Proposition 12 below), the converse implication fails even for pro strongly μ-measurable functions (see Example 13 below).

Proposition 12 (Bochner integral and a bounded operator). *Let $f : \Omega \to X$ be Bochner integrable, let Y be a Banach space and let $L : X \to Y$ be a bounded linear operator. Then $L \circ f$ is Bochner integrable and*

$$(B) \int_\Omega L \circ f \, \mathrm{d}\mu = L \left((B) \int_\Omega f \, \mathrm{d}\mu \right).$$

Remark: The preceding proposition shows that the Bochner integrability implies the Pettis one and, moreover, it can be use to compute the Bochner integral of a function: To this end it is necessary to show that the Bochner integral exists, its value can be then computed using suitable functionals or operators.

Example 13. *Let $\Omega = \mathbb{N}$, let Σ be the σ-algebra of all the subsets of $\mathbb{N}$, let μ be the counting measure and let $f : \Omega \to X$. Then:*

(a) *f is Bochner integrable if and only if the series $\sum_{n \in \mathbb{N}} f(n)$ is absolutely convergent. The Bochner integral then equals the sum of the series.*
(b) *If the series $\sum_{n \in \mathbb{N}} f(n)$ is unconditionally convergent, then f is Pettis integrable and its Pettis integral equals the sum of the series.*

Remark: In Example 13(b) the converse implication holds as well – if f is Pettis integrable, then the series $\sum_{n \in \mathbb{N}} f(n)$ is unconditionally convergent. The proof is more complicated, this statement is the content of Orlicz-Pettis theorem.