

VIII. Elements of vector integration

Convention: In this chapter we will use the following notation:

- (M, \mathcal{A}) is a fixed measurable space, i.e., M is a nonempty set and \mathcal{A} is a σ -algebra of subsets of M .
- (Ω, Σ, μ) is a fixed complete measure space, i.e., Ω is a nonempty set, Σ is a σ -algebra of subsets of Ω and μ is a non-negative σ -additive measure on Σ , which is moreover complete.
- X is a fixed Banach space over \mathbb{F} .

Remarks:

- (1) (Ω, Σ) is a special case of a measurable space. Therefore, whatever is stated below for (M, \mathcal{A}) , can be applied to (Ω, Σ) as well.
- (2) We do not a priori assume that μ is finite or σ -finite, even though these cases are the most important ones.

VIII.1 Measurability of vector-valued functions

Definition. Let $f : M \rightarrow X$ be a mapping. The function f is said to be

- **simple**, if its range is a finite set, i.e., if $f = \sum_{j=1}^k x_j \chi_{A_j}$, where $x_1, \dots, x_k \in X$ and A_1, \dots, A_k are nonempty pairwise disjoint subsets of M ;
- **simple measurable**, if it can be expressed as above and, moreover, $A_1, \dots, A_k \in \mathcal{A}$;
- **(strongly) \mathcal{A} -measurable** if there exists a sequence (u_n) of simple measurable functions pointwise converging to f (i.e., such that $\lim_{n \rightarrow \infty} \|u_n(t) - f(t)\| = 0$ for each $t \in M$);
- **Borel \mathcal{A} -measurable** if $f^{-1}(U) \in \mathcal{A}$ for each $U \subset X$ open;
- **weakly \mathcal{A} -measurable**, if $\varphi \circ f : M \rightarrow \mathbb{F}$ is (Borel) \mathcal{A} -measurable for each $\varphi \in X^*$.

Proposition 1.

- (a) *Simple functions, simple measurable functions, strongly \mathcal{A} -measurable functions and weakly \mathcal{A} -measurable functions form vector spaces.*
- (b) *Let (f_n) be a sequence of functions $f_n : M \rightarrow X$ pointwise converging to a function $f : M \rightarrow X$. If all the functions f_n are Borel \mathcal{A} -measurable (or weakly \mathcal{A} -measurable), the same holds for f .*
- (c) *Let $f : M \rightarrow X$ be a function. Then*

$$f \text{ strongly } \mathcal{A}\text{-measurable} \Rightarrow f \text{ Borel } \mathcal{A}\text{-measurable} \Rightarrow f \text{ weakly } \mathcal{A}\text{-measurable.}$$

For simple functions all the mentioned types of measurability coincide.

- (d) *If $f : M \rightarrow X$ is strongly \mathcal{A} -measurable, then $f(M)$ is a separable subset of X .*
- (e) *If $f : M \rightarrow X$ is Borel \mathcal{A} -measurable, then $t \mapsto \|f(t)\|$ is an \mathcal{A} -measurable (scalar-valued) function.*

Remarks:

- (1) Borel \mathcal{A} -measurable functions form a vector space if X is separable (by Theorem 3 below), in general they need not form a vector space.
- (2) The converse implications in (c) fail, see Examples 6 below.

Lemma 2. Let (f_n) be a sequence of strongly \mathcal{A} -measurable functions $f_n : M \rightarrow X$ pointwise converging to a function $f : M \rightarrow X$. Then f is strongly \mathcal{A} -measurable as well.

Theorem 3 (Pettis). Let $f : M \rightarrow X$ be a function. The following assertions are equivalent:

- (i) f is strongly \mathcal{A} -measurable.
- (ii) f is Borel \mathcal{A} -measurable and $f(M)$ is a separable subset of X .
- (iii) f is weakly \mathcal{A} -measurable and $f(M)$ is a separable subset of X .

Definition. Let $f : \Omega \rightarrow X$ be a mapping. The function f is said to be

- **(strongly) μ -measurable** if there exists a sequence (u_n) of simple measurable functions $u_n : \Omega \rightarrow X$ almost everywhere converging to f (i.e. such that $\lim_{n \rightarrow \infty} \|u_n(\omega) - f(\omega)\| = 0$ for almost all $\omega \in \Omega$);
- **Borel μ -measurable** (or **weakly μ -measurable**), if it is Borel Σ -measurable (or weakly Σ -measurable).

Remarks:

- (1) Let $f : \Omega \rightarrow X$ be a function. Then:

$$f \text{ strongly } \mu\text{-measurable} \Rightarrow f \text{ Borel } \mu\text{-measurable} \Rightarrow f \text{ weakly } \mu\text{-measurable}$$

- (2) If $f : \Omega \rightarrow X$ is (strongly) μ -measurable, then

$$\exists Y \subset\subset X \text{ separable} \exists N \in \Sigma : \mu(N) = 0 \ \& \ f(\Omega \setminus N) \subset Y.$$

A function satisfying this condition is called **essentially separably valued**.

Lemma 4. Let (f_n) be a sequence of strongly μ -measurable functions $f_n : M \rightarrow X$ almost everywhere converging to a function $f : M \rightarrow X$. Then f is strongly μ -measurable as well.

Theorem 5 (Pettis). Let $f : \Omega \rightarrow X$ be a function. The following assertions are equivalent:

- (i) f is strongly μ -measurable.
- (ii) f is Borel μ -measurable and essentially separably valued.
- (iii) f is weakly μ -measurable and essentially separably valued.

Examples 6.

- (1) Let $\Omega = [0, 1]$, let μ be the Lebesgue measure on $[0, 1]$ and let Σ be the σ -algebra of all the Lebesgue measurable subsets of $[0, 1]$. Consider the function $f : [0, 1] \rightarrow \ell^2([0, 1])$ defined by $f(t) = e_t$, $t \in [0, 1]$, where e_t denotes the respective canonical unit vector.

Then f is weakly μ -measurable, but fails to be essentially separably valued, hence it is not strongly μ -measurable. It is neither Borel μ -measurable.

- (2) Let (Ω, Σ, μ) and f be as in (1). Let moreover $h : [0, 1] \rightarrow [0, \infty)$ be any function. Then the function $h \cdot f$ is weakly μ -measurable as well. Further, for $t \in [0, 1]$ one has $\|h(t)f(t)\| = h(t)$. Therefore, if we choose h to be non-measurable, then $g = h \cdot f$ is weakly μ -measurable, but the function $t \mapsto \|g(t)\|$ is not measurable.

- (3) Let $\Omega = [0, 1]$, let Σ be the σ -algebra of all the subsets of $[0, 1]$, let μ be the counting measure and let f be as in (1). Then f is Borel μ -measurable, but fails to be essentially separably valued, thus it is not strongly μ -measurable.

Remark: The question, whether for a finite measure μ any Borel μ -measurable function is essentially separably valued (and hence strongly μ -measurable), is more complicated. The answer depends on additional axioms of the set theory.