VII.5 Tempered distributions

Convention: For $\boldsymbol{x} \in \mathbb{R}^d$ the symbol $\|\boldsymbol{x}\|$ denotes the Euclidean norm of $\boldsymbol{x}$.

Definition.

- By the **Schwartz space on** $\mathbb{R}^d$ we mean the space of functions

$$\mathscr{S}(\mathbb{R}^d) = \left\{ \begin{array}{ll} f \in \mathcal{C}^\infty(\mathbb{R}^d); & \text{the function } \boldsymbol{x} \mapsto (1+\|\boldsymbol{x}\|^2)^N D^\alpha f(\boldsymbol{x}) \\ & \text{is bounded on } \mathbb{R}^d \\ & \text{for each } N \in \mathbb{N}_0 \text{ and each multiindex } \alpha \end{array} \right\}.$$

  If d is fixed, we write just $\mathscr{S}$ instead of $\mathscr{S}(\mathbb{R}^d)$.
- For $f \in \mathscr{S}$ and $N \in \mathbb{N}_0$ we set

$$\begin{aligned} p_N(f) &= \sup_{\alpha \in \mathbb{N}_0^d, |\alpha| \le N} \left\| (1+\|\boldsymbol{x}\|^2)^N D^\alpha f(\boldsymbol{x}) \right\|_\infty \\ &= \sup\{ \left| (1+\|\boldsymbol{x}\|^2)^N D^\alpha f(\boldsymbol{x}) \right| ; \boldsymbol{x} \in \mathbb{R}^d, \alpha \in \mathbb{N}_0^d, |\alpha| \le N \}. \end{aligned}$$

  Then p_N is a norm on $\mathscr{S}$.

Proposition 17. *Let $d \in \mathbb{N}$.*

(a) *The Schwartz space $\mathscr{S}(\mathbb{R}^d)$ equipped with the sequence of norms (p_N) is a Fréchet space.*
(b) *$\mathscr{D}(\mathbb{R}^d)$ is a dense subspace of $\mathscr{S}(\mathbb{R}^d)$.*
(c) *If (φ_j) is a sequence in $\mathscr{D}(\mathbb{R}^d)$ converging in $\mathscr{D}(\mathbb{R}^d)$ to a function φ, then $\varphi_j \to \varphi$ in $\mathscr{S}(\mathbb{R}^d)$ as well.*

Definition. By a **tempered distribution** on $\mathbb{R}^d$ we mean a continuous linear functional on $\mathscr{S}(\mathbb{R}^d)$, i.e., a continuous linear mapping $\Lambda : \mathscr{S}(\mathbb{R}^d) \to \mathbb{F}$. The space of tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^d)$ or just by $\mathscr{S}'$.

Remark:

- If Λ is a tempered distribution on $\mathbb{R}^d$, its restriction to $\mathscr{D}(\mathbb{R}^d)$ is a distribution on $\mathbb{R}^d$ (by Proposition 17(c)).
- The space $\mathscr{S}'(\mathbb{R}^d)$ may be interpreted (due to the previous item and Proposition 17(b)) as a linear subspace of $\mathscr{D}'(\mathbb{R}^d)$.
- Using the interpretation from the previous item, a given distribution $\Lambda \in \mathscr{D}'(\mathbb{R}^d)$ is tempered if and only if there exists its continuous extension to $\mathscr{S}(\mathbb{R}^d)$.

Proposition 18 (a characterization of tempered distributions).

(a) *Let $\Lambda : \mathscr{S} \to \mathbb{F}$ be a linear mapping. Then Λ is a tempered distribution if and only if there are $N \in \mathbb{N}_0$ and $C > 0$ such that*

$$|\Lambda(\varphi)| \le C p_N(\varphi), \quad \varphi \in \mathscr{S}.$$

(b) *Let $\Lambda \in \mathscr{D}'(\mathbb{R}^d)$. Then Λ is tempered (in the sense of the previous remark) if and only if there are $N \in \mathbb{N}_0$ and $C > 0$ such that*

$$|\Lambda(\varphi)| \le C p_N(\varphi), \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Definition. Let $(\Lambda_n) \subset \mathscr{S}'$ be a sequence of tempered distributions on $\mathbb{R}^d$. We say that the sequence (Λ_n) **converges in** $\mathscr{S}'$ to tempered distribution Λ if it converges in the weak* topology, i.e., if $\Lambda_n(\varphi) \to \Lambda(\varphi)$ for any $\varphi \in \mathscr{S}$.

Theorem 19 (Banach-Steinhaus Theorem for tempered distribution). *Let (Λ_n) be a sequence of tempered distributions. If for each $\varphi \in \mathscr{S}$ there exists a finite limit $\lim_{n\to\infty} \Lambda_n(\varphi)$, then the limit mapping $\Lambda(\varphi) = \lim_{n\to\infty} \Lambda_n(\varphi)$, $\varphi \in \mathscr{S}$, is a tempered distribution.*

Proposition 20 (Examples of tempered distributions).

(a) *Any distribution with compact support is tempered.*

(b) *If $f \in L^p(\mathbb{R}^d)$ for some $p \in [1,\infty]$, then Λ_f is a tempered distribution and it is given by the formula*

$$\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi, \quad \varphi \in \mathscr{S}.$$

(c) *If f is a measurable function on $\mathbb{R}^d$ such that there exists a polynomial P on $\mathbb{R}^d$ satisfying $|f| \leq |P|$ on $\mathbb{R}^d$, then Λ_f is a tempered distribution and it is given by the formula from (b).*

(d) *If μ is a signed or complex measure on $\mathbb{R}^d$, then Λ_μ is a tempered distribution and it is given by the formula*

$$\Lambda_\mu(\varphi) = \int_{\mathbb{R}^d} \varphi \,\mathrm{d}\mu, \quad \varphi \in \mathscr{S}.$$

Remark: Since tempered distribution form a special case of distributions on $\mathbb{R}^d$, the operations with distributions defined in Sections VII.2-VII.4 (derivatives, multiplying by C^∞ functions, translation, reflexion, convolution with a function from $\mathscr{D}(\mathbb{R}^d)$) may be applied to tempered distributions as well, the respective result is a distribution. In some cases the result is even a tempered distribution.

Lemma 21 (continuity of operations on the Schwartz space). *The following mappings are continuous linear mappings of $\mathscr{S}$ to $\mathscr{S}$:*

- $f \mapsto P \cdot f$ *if P is a polynomial on $\mathbb{R}^d$,*
- $f \mapsto g \cdot f$ *if $g \in \mathscr{S}$,*
- $f \mapsto D^\alpha f$ *if α is a multiindex.*

Proposition 22 (operations with tempered distributions). *Let $\Lambda \in \mathscr{S}'(\mathbb{R}^d)$.*

(a) *Let α be a multiindex. Then $D^\alpha \Lambda \in \mathscr{S}'(\mathbb{R}^d)$. Moreover,*

$$D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|}\Lambda(D^\alpha \varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

(b) *Assume that $f \in \mathscr{S}(\mathbb{R}^d)$ or f is a polynomial on $\mathbb{R}^d$. Then $f\Lambda \in \mathscr{S}'(\mathbb{R}^d)$. Moreover,*

$$f\Lambda(\varphi) = \Lambda(f\varphi), \quad \varphi \in \mathscr{S}(\mathbb{R}^d).$$

(c) *Let $\boldsymbol{y} \in \mathbb{R}^d$. Then $\tau_{\boldsymbol{y}}\Lambda \in \mathscr{S}'(\mathbb{R}^d)$ and*

$$\tau_{\boldsymbol{y}}\Lambda(f) = \Lambda(\tau_{-\boldsymbol{y}} f), \quad f \in \mathscr{S}(\mathbb{R}^d).$$

(d) *$\check{\Lambda} \in \mathscr{S}'(\mathbb{R}^d)$. Moreover, $\check{\Lambda}(f) = \Lambda(\check{f})$ for $f \in \mathscr{S}(\mathbb{R}^d)$.*

Proposition 23. *Let (Λ_n) be a sequence in $\mathscr{S}'$ converging in $\mathscr{S}'$ to $\Lambda \in \mathscr{S}'$. Then:*

(a) *$D^\alpha \Lambda_n \to D^\alpha \Lambda$ in $\mathscr{S}'$ for each multiindex α.*

(b) *$f\Lambda_n \to f\Lambda$ if $f \in \mathscr{S}$* or *f is a polynomial on $\mathbb{R}^d$.*