# VII.3 A bit more on distributions

**Definition.** Let $\Omega \subset \mathbb{R}^d$ be an open set. We say that a sequence $(\Lambda_n)$ in $\mathscr{D}'(\Omega)$ **converges** to a distribution $\Lambda$, if it converges pointwise on $\mathscr{D}(\Omega)$, i.e., if $\Lambda_n(\varphi) \to \Lambda(\varphi)$ for each $\varphi \in \mathscr{D}(\Omega)$.

**Proposition 10** (on the convergence of distributions). *Let $\Omega \subset \mathbb{R}^d$ be an open set. Then:*

(a) *If $(\Lambda_n)$ is a sequence on distributions on $\Omega$ which converges to a distribution $\Lambda$, then*
- $D^\alpha \Lambda_n \to D^\alpha \Lambda$ *for each multiindex $\alpha$;*
- $f\Lambda_n \to f\Lambda$ *for any $f \in \mathcal{C}^\infty(\Omega)$.*

(b) *If $(f_n)$ is a sequence in $L^1_{\mathrm{loc}}(\Omega)$ converging in $L^1_{\mathrm{loc}}(\Omega)$ to a function $f$, i.e.,*
$$\int_K |f_n - f| \to 0 \text{ for any compact } K \subset \Omega,$$
*then $\Lambda_{f_n} \to \Lambda_f$.*

(c) *If $p \in [1, \infty]$ and $(f_n)$ is a sequence in $L^p(\Omega)$ converging in $L^p(\Omega)$ to a function $f$, then $\Lambda_{f_n} \to \Lambda_f$.*

(d) *If $(\varphi_n)$ is a sequence in $\mathscr{D}(\Omega)$ converging in $\mathscr{D}(\Omega)$ to a function $\varphi$, then $\Lambda_{\varphi_n} \to \Lambda_\varphi$.*

**Theorem 11** (Banach-Steinhaus theorem for distribution). *Let $(\Lambda_n)$ be a sequence of distributions on $\Omega$ such that the sequence $(\Lambda_n(\varphi))$ converges for any $\varphi \in \mathscr{D}(\Omega)$. If we set $\Lambda(\varphi) = \lim_n \Lambda_n(\varphi)$, $\varphi \in \mathscr{D}(\Omega)$, then $\Lambda \in \mathscr{D}'(\Omega)$.*

**Definition.** Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Lambda$ be a distribution on $\Omega$.

- Let $G \subset \Omega$ be open. $\Lambda$ is said to **vanish on** $G$ if $\Lambda(\varphi) = 0$ for any $\varphi \in \mathscr{D}(\Omega)$ such that $\operatorname{spt} \varphi \subset G$.
- **The support** of a distribution $\Lambda$ is the set
$$\begin{aligned}\operatorname{spt} \Lambda &= \Omega \setminus \bigcup \{G \subset \Omega \text{ open}; \Lambda \text{ vanishes on } G\} \\ &= \{x \in \Omega; \forall \varepsilon > 0 \exists \varphi \in \mathscr{D}(\Omega) : \operatorname{spt} \varphi \subset U(x, \varepsilon) \ \&\ \Lambda(\varphi) \neq 0\}\end{aligned}$$
- We say that $\Lambda$ **has compact support** if $\operatorname{spt} \Lambda$ is a compact subset of $\Omega$.

**Proposition 12** (on the support of a distribution). *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\Lambda$ be a distribution on $\Omega$. Then:*

(a) *If $\Lambda = \Lambda_f$ for $f \in L^1_{\mathrm{loc}}(\Omega)$, then $\operatorname{spt} \Lambda = \operatorname{spt} f$, where*
$$\operatorname{spt} f = \{\boldsymbol{x} \in \Omega; \lambda^d(\{\boldsymbol{y} \in U(\boldsymbol{x}, \varepsilon); f(\boldsymbol{y}) \neq 0\}) > 0 \text{ for each } \varepsilon > 0\}.$$
*If $f$ is continuous, this set coincides with $\operatorname{spt} f$ defined earlier.*

(b) *If $\Lambda = \Lambda_\mu$ for a measure $\mu$, then $\operatorname{spt} \Lambda = \operatorname{spt} \mu$, where*
$$\operatorname{spt} \mu = \Omega \setminus \{G \subset \Omega \text{ open}; \mu(A) = 0 \text{ for any } A \subset G \text{ Borel}\}.$$

(c) *If $\varphi \in \mathscr{D}(\Omega)$ is such that $\operatorname{spt} \varphi \cap \operatorname{spt} \Lambda = \emptyset$, then $\Lambda(\varphi) = 0$.*

(d) *If $\Lambda$ has compact support, then there exist $N \in \mathbb{N}_0$ and $C > 0$ such that*
$$|\Lambda(\varphi)| \leq C \|\varphi\|_N \text{ for each } \varphi \in \mathscr{D}(\Omega).$$
*In particular, $\Lambda$ is of finite order.*

(e) *$\operatorname{spt} \Lambda$ is a singleton $\{p\}$ if and only if there exist $N \in \mathbb{N}_0$ and numbers $c_\alpha$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$, not all zero such that*
$$\Lambda = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N} c_\alpha D^\alpha \Lambda_{\delta_p},$$
*i.e. there exist numbers $d_\alpha$, $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$, not all zero such that*
$$\Lambda(\varphi) = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N} d_\alpha D^\alpha \varphi(p), \quad \varphi \in \mathscr{D}(\Omega).$$