V. Locally convex spaces

Basic notation:

$\mathbb{R}$ ... the field of real numbers
$\mathbb{C}$ ... the field of complex numbers
$\mathbb{F}$ ... the field $\mathbb{R}$ or $\mathbb{C}$
If X is a vector space over $\mathbb{F}$, the zero vector is denoted by $\boldsymbol{o}$ (and sometimes by 0).
If X is a vector space over $\mathbb{F}$, $Y \subset\subset X$ means that Y is a subspace of X.

V.1 Locally convex topologies and their generating

Definition. A **topological vector space over** $\mathbb{F}$ is a pair $(X, \mathcal{T})$ where X is a vector space over $\mathbb{F}$ and $\mathcal{T}$ is a topology on X with the following two properties:

(1) The mapping $(x, y) \mapsto x + y$ is a continuous mapping of $X \times X$ into X.
(2) The mapping $(t, x) \mapsto tx$ is a continuous mapping of $\mathbb{F} \times X$ into X.

The term *topological vector space* will be abbreviated by **TVS**. If $(X, \mathcal{T})$ is moreover Hausdorff, we write **HTVS**.

The symbol $\mathcal{T}(\boldsymbol{o})$ will denote the family of all the neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$.

Definition. Let $(X, \mathcal{T})$ be a TVS. The space X is said to be **locally convex**, if there exists a base of neighborhoods of zero consisting of convex sets. The term *locally convex TVS* will be abbreviated by **LCS**, if it is moreover Hausdorff, then by **HLCS**.

Examples 1.

(1) *Let $(X, \|\cdot\|)$ is a normed linear space and let $\mathcal{T}$ be the topology generated by the norm (i.e., generated by the metric induced by the norm). Then $(X, \mathcal{T})$ is HLCS.*
(2) *Let Γ be any nonempty set. Then $\mathbb{F}^\Gamma$ is HLCS, if it is equipped by the product topology.*
(3) *The space $\mathcal{C}(\mathbb{R}, \mathbb{F})$ of continuous functions on $\mathbb{R}$ is HLCS, if it is equipped by the topology of locally uniform convergence. This topology is generated, for example, by the metric*

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \max\{|f(x) - g(x)|\,; x \in [-n, n]\}\}, \quad f, g \in \mathcal{C}(\mathbb{R}, \mathbb{F}).$$

(4) *Let $\Omega \subset \mathbb{C}$ be an open set. Then the space $H(\Omega)$ of holomorphic functions on Ω is HLCS, if it is equipped by the topology of locally uniform convergence. This topology is generated, for example, by the metric*

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, \max\{|f(z) - g(z)|\,; z \in K_n\}\}, \quad f, g \in H(\Omega),$$

where (K_n) is an exhausting sequence of compact subsets of Ω (i.e., sequence of compact subsets satisfying $K_n \subset \operatorname{Int} K_{n+1}$ for each $n \in \mathbb{N}$ and $\bigcup_n K_n = \Omega$).

(5) *Let* (Ω, Σ, μ) *be a measure space (where* μ *is a nonnegative measure) and* $p \in (0,1)$. *Then the space* $L^p(\Omega, \Sigma, \mu)$ *consisting of equivalence classes of measurable functions* $f : \Omega \to \mathbb{F}$ *satisfying* $\int_\Omega |f|^p \,\mathrm{d}\mu < \infty$ *is HTVS, if it is equipped by the topology generated by the metric*

$$\rho(f,g) = \int_\Omega |f-g|^p \,\mathrm{d}\mu, \qquad f,g \in L^p(\Omega, \Sigma, \mu).$$

If, for example, $\Omega = [0,1]$ *and* μ *is the Lebesgue measure or* $\Omega = \mathbb{N}$ *and* μ *is the counting measure, then this space fails to be locally convex.*

Remark: In the sequel we will deal only with locally convex spaces. As for the general topological vector spaces – some areas of the theory are completely analogous, some areas are similar with substantially more complicated proofs and some areas are completely different. We will point out some similarities and differences in remarks and few examples.

Observation: If $(X, \mathcal{T})$ is LCS, then $\mathcal{T}$ is translation invariant. I.e., if $A \subset X$ and $x \in X$, then A is open of and only if $x + A$ is open. It follows that $A \subset X$ is a neighborhood of $x \in X$ if and only if $-x + A \in \mathcal{T}(\boldsymbol{o})$. Therefore the family $\mathcal{T}(\boldsymbol{o})$ uniquely determines the topology $\mathcal{T}$.

Definition. Let X be a vector space over $\mathbb{F}$ and $A \subset X$. We say that the set A is

- **convex**, if $tx + (1-t)y \in A$ whenever $x, y \in A$ and $t \in [0,1]$;
- **symmetric**, if $A = -A$;
- **balanced**, if $\alpha A \subset A$ whenever $\alpha \in \mathbb{F}$ is such that $|\alpha| \le 1$;
- **absolutely convex**, if it is convex and balanced;
- **absorbing**, if for each $x \in X$ there exists $t > 0$ such that $\{sx; s \in [0,t]\} \subset A$.

Definition. Let X be a vector space over $\mathbb{F}$ and $A \subset X$. By the **convex hull** (**balanced hull**, **absolutely convex hull**) of A we mean the smallest convex (balanced, absolutely convex) set containing A. This set is denoted by $\mathrm{co}(A)$ ($\mathrm{b}(A)$, $\mathrm{aco}(A)$, respectively).

Proposition 2. *Let* X *be a vector space over* $\mathbb{F}$ *and* $A \subset X$.

(a) *If* $\mathbb{F} = \mathbb{R}$, *then* A *is absolutely convex, if and only if it is convex and symmetric.*
(b) $\mathrm{co}(A) = \{t_1 x_1 + \cdots + t_k x_k; x_1, \dots, x_k \in A, t_1, \dots, t_k \ge 0, t_1 + \cdots + t_k = 1\}$.
(c) $\mathrm{b}(A) = \{\alpha x; x \in A, \alpha \in \mathbb{F}, |\alpha| \le 1\}$.
(d) $\mathrm{aco}(A) = \mathrm{co}(\mathrm{b}(A))$.
(e) A *is convex if and only if* $(s+t)A = sA + tA$ *for any* $s, t \in (0, \infty)$.

Proposition 3. *Let* $(X, \mathcal{T})$ *be a LCS and* $U \in \mathcal{T}(\boldsymbol{o})$. *Then:*

(i) U *is absorbing.*
(ii) *There exists* $V \in \mathcal{T}(\boldsymbol{o})$ *such that* $V + V \subset U$.
(iii) *There exists* $V \in \mathcal{T}(\boldsymbol{o})$ *open and absolutely convex such that* $V \subset U$.

Theorem 4.

(1) *Let $(X, \mathcal{T})$ be a LCS. Then there exists $\mathcal{U}$, a base of neighborhoods of $\boldsymbol{o}$ with the following properties:*
   (i) *The elements of $\mathcal{U}$ are absorbing, open and absolutely convex.*
   (ii) *For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $2V \subset U$.*

   If X is moreover Hausdorff, then $\bigcap \mathcal{U} = \{\boldsymbol{o}\}$.

(2) *Conversely, let X be a vector space and $\mathcal{U}$ a family of subsets of X with the following properties:*
   (i) *The elements of $\mathcal{U}$ are absorbing and absolutely convex.*
   (ii) *For any $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $2V \subset U$.*
   (iii) *For any $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W \subset U \cap V$.*

   Then there exists a unique topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ is a LCS and $\mathcal{U}$ is a base of neighborhoods of $\boldsymbol{o}$. Further, if $\bigcap \mathcal{U} = \{\boldsymbol{o}\}$, then $\mathcal{T}$ is Hausdorff.

Theorem 5 (on the topology generated by a family of seminorms). *Let X be a vector space and let $\mathcal{P}$ be a nonempty family of seminorms on X. Then there exists a unique topology $\mathcal{T}$ na X such that $(X, \mathcal{T})$ is LCS and the family*

$$\Big\{\{x \in X; p_1(x) < c_1, \dots, p_k(x) < c_k\}; p_1, \dots, p_k \in \mathcal{P}, c_1, \dots, c_k > 0\Big\}$$

is a base of neighborhoods of $\boldsymbol{o}$ in $(X, \mathcal{T})$. The topology $\mathcal{T}$ is Hausdorff if and only if for each $x \in X \setminus \{\boldsymbol{o}\}$ there exists $p \in \mathcal{P}$ such that $p(x) > 0$.

Definition. The topology $\mathcal{T}$ from Theorem 5 is called **the topology generated by the family of seminorms** $\mathcal{P}$.

Examples 6.

(1) *If $(X, \|\cdot\|)$ is a normed linear space, then $\|\cdot\|$ is a seminorm on X. The topology generated by the norm coincides with the topology generated by the one-element family of seminorms $\{\|\cdot\|\}$.*

(2) *The product topology on $\mathbb{F}^\Gamma$ (from Example 1(2)) coincides with the topology generated by the family of seminorms $\{p_\gamma; \gamma \in \Gamma\}$, where*

$$p_\gamma(f) = |f(\gamma)|, \quad f \in \mathbb{F}^\Gamma.$$

(3) *The topology of locally uniform convergence on $\mathcal{C}(\mathbb{R}, \mathbb{F})$ from Example 1(3) coincides with the topology generated by the sequence of seminorms $(p_n)_{n \in \mathbb{N}}$, where*

$$p_n(f) = \sup\{|f(x)|; x \in [-n, n]\}, \quad f \in \mathcal{C}(\mathbb{R}, \mathbb{F}).$$

(4) *Let T be a Hausdorff topological space and let $\mathcal{C}(T, \mathbb{F})$ denote the space of all the continuous functions on T. Then the family of seminorms*

$$\mathcal{P} = \{p_K; K \subset T \text{ compact}\}, \text{ where } p_K(f) = \sup\{|f(x)|; x \in K\}, \quad f \in \mathcal{C}(T, \mathbb{F}),$$

generates the topology of uniform convergence on compact subsets of T. If T is locally compact, it is the topology of locally uniform convergence.

Definition. Let X be a vector space and let $A \subset X$ be a convex absorbing set. By the **Minkowski functional** of the set A we mean the function defined by the formula

$$p_A(x) = \inf\{\lambda > 0; x \in \lambda A\}, \qquad x \in X.$$

Lemma 7. *Let X be a LCS and let $A \subset X$ be a convex set. If $x \in \overline{A}$ and $y \in \operatorname{Int} A$, then $\{tx + (1-t)y; t \in [0,1)\} \subset \operatorname{Int} A$.*

Proposition 8 (on the Minkowski functional of a convex neighborhood of zero). *Let X be a LCS amd let $A \subset X$ be a convex neighborhood of $\boldsymbol{o}$. Then:*

- *p_A is continuous on X.*
- *$\operatorname{Int} A = \{x \in X; p_A(x) < 1\}$.*
- *$\overline{A} = \{x \in X; p_A(x) \leq 1\}$.*
- *$p_A = p_{\overline{A}} = p_{\operatorname{Int} A}$.*

Corollary 9. *Any LCS is completely regular. Any HLCS is Tychonoff.*

Theorem 10 (on generating of locally convex topologies). *Let $(X, \mathcal{T})$ be a LCS. Let $\mathcal{P}_{\mathcal{T}}$ be the family of all continuous seminorms on $(X, \mathcal{T})$. Then the topology generated by the family $\mathcal{P}_{\mathcal{T}}$ equals $\mathcal{T}$.*

Proposition 11. *Let X be a vector space.*

1. *If p is a seminorm on X, then the set $A = \{x \in X; p(x) < 1\}$ is absolutely convex, absorbing and satisfies $p = p_A$.*
2. *Let p, q be two seminorms on X. Then $p \leq q$ if and only if*
$$\{x \in X; p(x) < 1\} \supset \{x \in X; q(x) < 1\}.$$
3. *Let $\mathcal{P}$ be a nonempty family of seminorms on X and let $\mathcal{T}$ be the topology generated by the family $\mathcal{P}$. Let p be a seminorm on X. Then p is $\mathcal{T}$-continuous if and only if there exist $p_1, \dots, p_k \in \mathcal{P}$ and $c > 0$ such that $p \leq c \cdot \max\{p_1, \dots, p_k\}$.*

Remark: For general TVS the following statements from this section are valid:

- Observation after Examples 1 (no change needed).
- Proposition 3, if we replace 'absolutely convex' by 'balanced' in assertion (iii).
- Theorem 4, if we replace everywhere 'absolutely convex' by 'balanced' and '$2V \subset U$' by '$V + V \subset U$'.
- Lemma 7 and Proposition 8 (no change needed).
- Corollary 9 (no change needed, but the proof is substantially more complicated).