Let X be a LCS and $U \subset X$ a nbhd of 0
Then
(a) $U^0 = \{ f \in X^* ; \ \forall x \in U : |f(x)| \leq 1 \}$ is a weak*-compact subset of X^*

(b) If X is moreover separable, then U^0 is metrizable in the weak*-topology.

<u>Proof</u> (a) Consider

$$T : X^* \longrightarrow \mathbb{F}^U \quad \text{defined by}$$

$$T(f)(x) = f(x), \ f \in X^*, x \in U,$$
i.e. $T(f) = f|U$

Then T is a homeomorphism of (X^*, w^*) onto $\mathbb{F}^U$

Γ • T is one-to-one: $T(f) = T(g) \Rightarrow f|U = g|U$.
Since f, g are linear and U is absorbing, necessarily $f = g$

• T is continuous (on $\mathbb{F}^U$ we consider the topology of pw convergence):

$x \in U$ fixed $\Rightarrow f \mapsto T(f)(x) = f(x)$ is w^*-cts by the definition of the w^*-topology
so, by Prop. VI.1(6) T is cts

• T^{-1} is cts on $T(X^*)$

Fix $x \in X$. Since U is absorbing, there is $t > 0$ with $tx \in U$
If $g = T(f) \in T(X^*)$, then
$T^{-1}(g)(x) = f(x) = \frac{1}{t} f(tx) = \frac{1}{t} g(tx)$, so
$g \mapsto T^{-1}(g)(x)$ is cts. $\rfloor$
by Prop VI.1(6) we deduce that T^{-1} is cts

Moreover,

$$T(U^\circ) = \left\{ F \in \mathbb{F}^U \;;\; \begin{array}{l} \forall x \in U : |F(x)| \leq 1 \\ \& \\ \forall \alpha, \beta \in \mathbb{F}\ \forall x, y \in U : \alpha x + \beta y \in U \Rightarrow \\ \qquad \Rightarrow F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \end{array} \right\}$$

Γ "$\subset$" clear

"$\supset$" : Let F be in the set on the RHS
We will define $f : X \to \mathbb{F}$ as follows:

Let $x \in X$. Find $\alpha > 0$ s.t. $\alpha x \in U$
and set $f(x) = \frac{1}{\alpha} F(\alpha x)$.

- $F(0) = 0$ Γ $F(0+0) = F(0) + F(0)$ $\lrcorner$

- f is well defined:
  $x \in X$, $\alpha, \beta > 0$ $\alpha x, \beta x \in U$

  Then $\frac{1}{\alpha}(\alpha x) - \frac{1}{\beta}(\beta x) = 0 \in U$,

  so $0 = F(0) = F\left(\frac{1}{\alpha}(\alpha x) - \frac{1}{\beta}(\beta x)\right) =$

  $= \frac{1}{\alpha} F(\alpha x) - \frac{1}{\beta} F(\beta x)$,

  hence $\frac{1}{\alpha} F(\alpha x) = \frac{1}{\beta} F(\beta x)$

- f is linear: $x, y \in X$, $\alpha, \beta \in \mathbb{F}$.

U absorbing $\Rightarrow \exists\, t > 0$ s.t. $tx, ty, t(\alpha x + \beta y) \in U$

Then $f(\alpha x + \beta y) = \frac{1}{t} F(t(\alpha x + \beta y)) = \frac{1}{t} F(\alpha \cdot (tx) + \beta \cdot (ty))$

$= \frac{1}{t}(\alpha F(tx) + \beta F(ty)) = \alpha \cdot \frac{1}{t} F(tx) + \beta \cdot \frac{1}{t} F(ty) =$

$= \alpha f(x) + \beta f(y)$

- f is cts, as $\forall x \in U : |f(x)| \leq 1$
(and U is a nbhd of 0),

hence also $f \in U^0$, so $F = T(f) \in T(U^0)$. ⌟

So, $T(U^0)$ is a closed subset of

$\{\lambda \in \mathbb{F}, |\lambda| \leq 1\}^U$, which is compact

by Tychonoff theorem. So, U^0 is w^*-compact.

(5) Let X be moreover separable. Let $D \subset X$ be a ctble dense subset.

Then $\sigma(X^*, D)$ is Hausdorff

$\ulcorner$ D separates points of X^*:
$f \in X^*$, $f|_D = 0 \Rightarrow f \equiv 0$ as D is dense ⌟

and $\sigma(X^*, D)$ is metrizable

$\ulcorner$ D ctble $\Rightarrow \sigma(X^*, D)$ generated by a ctble family of seminorms
and use Theorem V.22 ⌟

on U^0: $\sigma(X^*, D)$ is a weaker Hausdorff topology than $\sigma(X^*, X)$
U^0 $\sigma(X^*, X)$ compact $\Rightarrow \sigma(X^*, X) = \sigma(X^*, D)$ on U^0.