Let X be a LCS and $A \subset X$ a $\sigma(X, X^*)$-bdd set. Then A is bdd also in the original topology of X.

Proof ① A is $\sigma(X, X^*)$-bdd $\Rightarrow \forall f \in X^* : f$ is bdd on A.
Therefore, in case X is a normed space, this theorem is a corollary to uniform bddness principle (see Corollary $\mathbb{I}.29$)

② Let $A \subset X$ be $\sigma(X, X^*)$-bdd.
To prove A is bdd, it is enough to show that any cts seminorm on X is bdd on A $\binom{\text{Lemma}}{V.15}$
So, let p be any cts seminorm on X

Set $Y := \{x \in X; \; p(x) = 0\}$
Then Y is a closed linear subspace of X
⌈ $\cdot$ Y is closed as p is cts
  $\cdot$ $0 \in Y$ as $p(0) = 0$
  $\cdot$ $x \in Y, \lambda \in \mathbb{F} \Rightarrow p(\lambda x) = |\lambda| p(x) = 0 \Rightarrow \lambda x \in Y$
  $\cdot$ $x, y \in Y \Rightarrow 0 \le p(x+y) \le p(x) + p(y) = 0 \Rightarrow x+y \in Y$⌋

Let X/Y be the quotient in the linearly-algebraic sense and $q : X \to X/Y$ be the canonical quotient map.

Define a norm $\|\cdot\|$ on X/Y by $\|q(x)\| = p(x)$, $x \in X$
⌈ well defined: $q(x) = q(y) \Rightarrow q(x-y) = 0 \Rightarrow x-y \in Y$, hence $p(x-y) = 0$. so,
$$p(x) \le p(x-y) + p(y) = p(y) \left.\right\} \Rightarrow$$
$$p(y) \le p(y-x) + p(x) = p(x) \quad \left.\right\} p(x) = p(y)$$

it is a norm: $\|0\| = \|q(0)\| = p(0) = 0$
$\|q(x)\| = 0 \Rightarrow p(x) = 0 \Rightarrow x \in Y \Rightarrow q(x) = 0$
$\|\lambda \cdot q(x)\| = \|q(\lambda x)\| = p(\lambda x) = |\lambda| p(x) =$
$\qquad = |\lambda| \cdot \|q(x)\|$
$\|q(x) + q(y)\| = \|q(x+y)\| = p(x+y) \le$
$\qquad \le p(x) + p(y) = \|q(x)\| + \|q(y)\|$ ⌋

q is cts $X \longrightarrow (+/\mu, \| \cdot \|)$ as

$\| q(+) \| \leq p(+)$ (in fact "$=$") and p is
a cts seminorm on X (by Lemma V.13)

Further, the set $q(A)$ is weakly bdd in $(+/\mu, \| \cdot \|)$

$\lceil f \in (+/\mu)^* \Rightarrow f \circ q \in X^* \Rightarrow (f \circ q)(A)$ is bdd,
and, finally, $f(q(A)) = (f \circ q)(A)$. $\rfloor$

So, by ① (the case of normed spaces), $q(A)$ is

norm-bdd in X/μ. i.e., $\exists C > 0 \; \forall x \in A : \| q(+) \| \leq C$

But $\| q(+) \| = p(+)$. Hence, $p \leq C$ on A and the proof
is finished.

Remark on the idea of the proof:
- The statement is valid for normed spaces by uniform bddness principle
- The statement is valid for semi-normed spaces $\lceil$ this case may be reduced
    to the normed case by taking
    the quotient $\rfloor$

- The statement is valid for LCS $\lceil$ as LCS are defined using seminorms
    and bddness may be tested using seminorms $\rfloor$