Let X and Y be Fréchet spaces and $T : X \to Y$ be a continuous linear mapping s.t. $T(X) = Y$. Then T is an open mapping

Proof: ① It is enough to prove that $\forall U \subset X$, neighborhood of 0 : TU is a nbhd of 0 in Y

② We will show : $\forall U \subset X$ nbhd of 0 : $\overline{TU}$ is a nbhd of 0 in Y

$\ulcorner$ U nbhd of 0 in X $\Rightarrow \exists V \subset U$ absolutely convex nbhd of 0

V absorbing $\Rightarrow \bigcup_{n=1}^{\infty} nV = X$, so $\bigcup_{n=1}^{\infty} T(nV) = Y$

Y is a completely metrizable space, so by Baire category thm $\exists n \in \mathbb{N}$: $\overline{T(nV)}$ has nonempty interior.

But $\overline{T(nV)} = \overline{nT(V)} = n \cdot \overline{T(V)}$

(since T is linear, continuous and $y \mapsto ny$ is a homeomorphism of Y)

So, $\operatorname{Int} \overline{T(nV)} = \operatorname{Int} n \cdot \overline{T(V)} = n \cdot \operatorname{Int} \overline{TV}$

In particular, $\operatorname{Int} \overline{TV} \neq \emptyset$.

So, there is W, an absol. convex nbhd of 0 in Y and $y \in Y$ s.t. $y + W \subset \overline{TV}$.

$\overline{TV}$ absolutely convex $\Rightarrow$ $\underset{-y+W}{\underbrace{-(y+W)}} \subset \overline{TV}$

and so $W \subset \overline{TV}$ $\left(w \in W \Rightarrow w = \frac{1}{2}\left(\underset{\in \overline{TV}}{(y+w)} + \underset{\in \overline{TV}}{(-y+w)}\right) \right.$ and $\overline{TV}$ is convex)

Thus $\overline{TV}$ is a nbhd of 0 and hence also $\overline{TU} \supset \overline{TV}$ is a nbhd of 0 in Y $\lrcorner$

(3) We will prove: $\forall U \subset X$ nbhd of 0: TU is a nbhd of 0
Fix a complete translation invariant metric ρ on X and set

$$U_n = \{x \in X ; \rho(x,0) < \tfrac{1}{2^n}\}, \quad n = 0,1,2,\dots$$

Then. (U_n) is a base of nbhds of 0 in X, it suffices to prove that TU_n is a nbhd of 0 for each n

Let us prove it for $n=0$, i.e. TU_0 is a nbhd of 0
(the general case is the same, or, replace ρ by $2^n \rho$):

By (2) we know that $\forall n \in \mathbb{N}$ $\overline{TU_n}$ is a nbhd of 0.
We will be done if we show $TU_0 \supset \overline{TU_1}$

To this end fix $y \in \overline{TU_1}$

Since $\overline{TU_2}$ is a nbhd of 0, we have $(y - \overline{TU_2}) \cap TU_1 \neq \emptyset$. So, there is $x_1 \in U_1$

s.t. $y - Tx_1 \in \overline{TU_2}$

Since $\overline{TU_3}$ is a nbhd of 0, we have $(y - Tx_1 - \overline{TU_3}) \cap TU_2 \neq \emptyset$

Hence there is $x_2 \in U_2$ s.t. $y - Tx_1 - Tx_2 \in \overline{TU_3}$

By induction we can find $x_n \in U_n$ for $n \in \mathbb{N}$

s.t. $y - Tx_1 - Tx_2 - \dots - Tx_n \in \overline{TU_{n+1}}$, $n \in \mathbb{N}$

Set $x := \sum_{n=1}^{\infty} x_n = \lim_{N \to \infty} \sum_{n=1}^{N} x_n$

This is well-defined, since (X,ρ) is complete and the sum is conv.

Indeed, if $m > n$, then

$$\rho\left(\sum_{k=1}^{m} x_k, \sum_{k=1}^{n} x_k\right) \overset{\Delta}{\leq} \sum_{\ell=n+1}^{m} \rho\left(\sum_{k=1}^{\ell} x_k, \sum_{k=1}^{\ell-1} x_k\right)$$

$$\overset{*}{=} \sum_{\ell=n+1}^{m} \rho(x_\ell, 0) < \sum_{\ell=n+1}^{m} 2^{-\ell} < 2^{-n},$$

where we used translation invariance of ρ (*) and the triangle inequality (Δ)

Moreover, $x \in U_0$, since

$$\rho(x, 0) = \lim_{n\to\infty} \rho\left(\sum_{k=1}^{n} x_k, 0\right) \overset{\Delta}{\leq} \lim_{n\to\infty} \sum_{k=1}^{n} \rho\left(\sum_{\ell=1}^{k} x_\ell, \overbrace{\sum_{\ell=1}^{k-1} x_\ell}^{0 \text{ if } k=1}\right)$$

$$\overset{*}{=} \lim_{n\to\infty} \sum_{k=1}^{n} \rho(x_k, 0) = \sum_{k=1}^{\infty} \rho(x_k, 0) \underset{x_k \in U_k}{<} \sum_{k=1}^{\infty} 2^{-k} = 1$$

Finally, $Tx = y$:

$$y - Tx = \lim_{n\to\infty} (y - Tx_1 - \dots - Tx_n)$$

$$y - Tx_1 - \dots - Tx_n \in \overline{TU_{n+1}} \subset \overline{TU_k} \text{ for } n+1 > k$$

So, for each $k \in \mathbb{N}$ $y - Tx \in \overline{TU_k}$, hence $y - Tx \in \bigcap_{k=1}^{\infty} \overline{TU_k}$

To finish, observe that $\bigcap_{k=1}^{\infty} \overline{TU_k} = \{0\}$

$\ulcorner y \in Y, y \neq 0 \overset{Y \text{ metrizable}}{\Rightarrow} \exists\, V$, nbhd of 0 in Y s.t. $y \notin \overline{V}$

T cts $\Rightarrow \exists\, k$ s.t. $T(U_k) \subset V$

Then $\overline{T(U_k)} \subset \overline{V}$, hence $y \notin \overline{TU_k}$ $\lrcorner$