$\mathbb{F}^{\mathbb{N}}$ is a Frechet space

(1) $p_n((t_k)_{k=1}^{\infty}) := \max\{|t_1|, \dots, |t_n|\} \Rightarrow (p_n)_{n=1}^{\infty}$ is a generating family of seminorms, moreover $p_1 \leq p_2 \leq p_3 \leq \dots$

$$\rho((t_k), (y_k)) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n((t_k)-(y_k))\}$$

The ρ is a translation invariant metric generating the topology of $\mathbb{F}^{\mathbb{N}}$ (by Proposition V.21 (2))

(2) We will prove that ρ is complete

Let $(x^k)_{k=1}^{\infty}$ be a ρ-cauchy sequence.

Then $\forall n$: (x^k) is cauchy in the seminorm p_n (by Proposition V.21 (2) (s))

Note that $p_n(x) = \|(x_1, \dots, x_n)\|_{\infty}$. It follows that

$\forall n$: $((x_1^k, \dots, x_n^k))_{k=1}^{\infty}$ is $\|\cdot\|_{\infty}$-cauchy in $\mathbb{F}^n$.

So, by completeness of $(\mathbb{F}^n, \|\ \|_{\infty})$, the sequence $((x_1^k, \dots, x_n^k))_{k=1}^{\infty}$ converges in $\mathbb{F}^n$ to some $(y_1^n, \dots, y_n^n) \in \mathbb{F}^n$

Since the convergence in $\mathbb{F}^n$ is coordinate-wise, we get

$$y_j^n = y_j^m \quad \text{for } j \le n < m.$$

So, we get one sequence $y = (y_j)_{j=1}^{\infty} \in \mathbb{F}^{\mathbb{N}}$ s.t.

$x_j^k \xrightarrow{k\to\infty} y_j$ for each $j \in \mathbb{N}$, i.e., $x^k \to y$ in $\mathbb{F}^{\mathbb{N}}$.

$\mathcal{C}(\mathbb{R})$ is a Frechet space:

① $P_n(f) = \max\{|f(x)|;\ x \in [-n,n]\}$, $n \in \mathbb{N}$
... it is a generating sequence of seminorms s.t. $P_1 \le P_2 \le P_3 \le \dots$

$$\rho(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, P_n(f-g)\}$$

is a translation invariant metric generating the topology of $\mathcal{C}(\mathbb{R})$ (by Proposition V.21 (2))

② ρ is complete:

Let $(f_k)_{k=1}^{\infty}$ be a ρ-cauchy sequence.

Then $\forall n \in \mathbb{N}$ (f_k) is cauchy in P_n (Prop. V. 21 (2) (c))

Since $P_n(f) = \| f\restriction_{[-n,n]} \|_{\infty}$, we get that

$(f_k\restriction_{[-n,n]})_{k=1}^{\infty}$ is uniformly cauchy on $[-n,n]$

Hence, it uniformly converges to a function

$g_n \in \mathcal{C}([-n,n])$.

So, $\forall n \in \mathbb{N}$: $\quad f_k \restriction_{[-n,n]} \rightrightarrows g_n$ on $[-n,n]$

Since the uniform convergence implies the pointwise convergence, we deduce that

$g_n = g_m \restriction_{[-n,n]}$ for $m > n$

Therefore, the function

$$g(x) = g_n(x),\ x \in [-n,n],\ n \in \mathbb{N}$$

is a well-defined continuous function on $\mathbb{R}$ and

$\forall n \in \mathbb{N}\ p_n(f_k - g) \to 0$, hence $f_k \to g$ in ρ. (Prop. V.21 (2)(4))

$H(\Omega)$ is a Fréchet space: ① Let (K_n) be an exhausting sequence of compacts
($\Omega \subset \mathbb{C}$ open) $\quad (K_n \subset \operatorname{Int} K_{n+1},\ \bigcup_n K_n = \Omega)$

$p_n(f) = \sup_{z \in K_n} |f(z)| \quad \Rightarrow \quad (p_n)$ is a generating sequence of seminorms.

$p_1 \leq p_2 \leq p_3 \leq \dots$

$\Rightarrow \rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \min\{1, p_n(f-g)\}$ is a translation invariant metric generating the topology of $H(\Omega)$ (Prop. V.21 (2))

② ρ is complete: Let (f_k) be a ρ-Cauchy sequence

Then $\forall n \in \mathbb{N}$: (f_k) is p_n-Cauchy (Prop. V.21 (2) (3))

Since $p_n(f) = \| f \restriction_{K_n} \|_\infty$, we deduce that $(f_k \restriction_{K_n})_{k=1}^{\infty}$ is $\| \|_\infty$-Cauchy, so there is $g_n \in C(K_n)$ s.t. $f_k \restriction_{K_n} \overset{k\to\infty}{\rightrightarrows} g_n$

Since the uniform convergence imply the pointwise one, we get $\quad g_m \restriction_{K_n} = g_n$ for $n \leq m$.

Hence, we have one function $g : \Omega \to \mathbb{C}$ s.t.

$\forall n \in \mathbb{N}$: $f_k \restriction_{K_n} \rightrightarrows g \restriction_{K_n}$. Thus $f_k \overset{loc}{\rightrightarrows} g$ on Ω.

By Weierstrass theorem we deduce that g is holomorphic, i.e. $g \in H(\Omega)$.

Since $\forall n$: $p_n(f_k - g) \xrightarrow{k} 0$, by Prop. V.21(2)(a) we conclude $\rho(f_k, g) \to 0$, i.e. $f_k \to g$ in $H(\Omega)$.

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