Let X be a HLCS admitting a totally bdd nbhd of 0. Then $\dim X < \infty$

Pf: Let U be an absolutely convex totally bdd open nbhd of 0

Then $\frac{1}{2}U$ is also a nbhd of 0, so, there is

$F \subset X$ finite with $U \subset F + \frac{1}{2}U$

Set $Y := \operatorname{span} F$. We claim that $Y = X$

Claim: $\forall n \in \mathbb{N}: U \subset Y + 2^{-n}U$

By induction: $n=1$ -- follows by ---

$n \to n+1$ Suppose $U \subset Y + 2^{-n}U$

Then: $U \subset Y + 2^{-n}U = Y + 2^{-n+1}(\frac{1}{2}U) \subset$

$\subset Y + 2^{-n+1}(\frac{1}{2}(Y + \frac{1}{2}U)) =$

$= Y + 2^{-n+1}(Y + \frac{1}{4}U) = Y + 2^{-n+1}Y + 2^{-n-1}U$

$= Y + 2^{-n-1}U$

here we used that Y is a linear subsp.

If $Y \neq X$, then $\exists x \in X \setminus Y$. Since U is absorbing, $\exists \varepsilon > 0$ s.t. $\varepsilon x \in U$. So $U \setminus Y \neq \emptyset$

Fix $x \in U \setminus Y$

X Hausdorff, $\dim Y < \infty \Rightarrow Y$ is closed. Hence there is V, an absolutely convex nbhd of 0 s.t. $x + V \subset U \setminus Y$

Since U is totally bdd, it is also bdd, so $\exists n \in \mathbb{N}: U \subset 2^n V$, i.e. $\frac{1}{2^n}U \subset V$

It follows that $x + \frac{1}{2^n}U \cap Y = \emptyset \Rightarrow x \notin Y + \frac{1}{2^n}U$

So, $x \in U \setminus (Y + \frac{1}{2^n}U)$, a contradiction with the Claim.