Proposition

Let X be a HLCS of finite dimension

Then:

(a) $\forall\, Y$ LCS $\forall\, L: X \to Y$: L is continuous

(b) X is isomorphic to $\mathbb{F}^n$, where $n = \dim X$

Proof: If $\dim X = 0$, i.e., $X = \{0\}$, it is trivial

Assume $n := \dim X \in \mathbb{N}$

Fix a basis $x_1, \dots, x_n$ of X

Define $T : (\mathbb{F}^n, \|\cdot\|_2) \to X$ by

$$T(\lambda_1, \dots, \lambda_n) = \lambda_1 x_1 + \dots + \lambda_n x_n$$

- T is clearly a linear bijection of $\mathbb{F}^n$ onto X (since $x_1, \dots, x_n$ is a basis)

- T is continuous:

$\boxed{1}$ The mapping $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_j$ is continuous ($\mathbb{F}^n \to \mathbb{F}$) for each j

(known from basic calculus)

$\boxed{2}$ $\forall\, x \in X$: the mapping $\lambda \mapsto \lambda \cdot x$ is continuous $\mathbb{F} \to X$

(this follows from the continuity of multiplication in TVS)

$\boxed{3}$ By composing: $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_j x_j$

is continuous $\mathbb{F}^n \to X$

Hence, T is the sum of n continuous mappings
$\mathbb{F}^n \to X$
It is enough to show that the sum of two continuous mappings is continuous and use mathematical induction.

Γ Ω topological space, X LCS
$f_1, f_2 : \Omega \to X$ continuous
$\Rightarrow f_1 + f_2$ is continuous

Pf: $t \in \Omega$ arbitrary, let $G \subset X$ be open
s.t. $f_1(t) + f_2(t) \in G$

$\Rightarrow \exists\, U$ nbhd of 0 : $f_1(t) + f_2(t) + U \subset G$

$\exists\, V$ nbhd of 0 : $V + V \subset U$

f_1 cts at $t \Rightarrow \exists\, W_1$ open in Ω, $t \in W_1$
$f_1(W_1) \subset f_1(t) + V$

f_2 cts at $t \Rightarrow \exists\, W_2$ open in Ω, $t \in W_2$
$f_2(W_2) \subset f_2(t) + V$

$W := W_1 \cap W_2$ ... open in Ω, $t \in W$

$s \in W \Rightarrow f_1(s) + f_2(s) \in f_1(t) + V + f_2(t) + V$

$\subset f_1(t) + f_2(t) + U \subset G$

$\perp$

This completes the proof that T is cts

T^{-1} is cts as well:

$S_{\mathbb{F}_n}$ ... the sphere of $\mathbb{F}^n$ is compact in $\mathbb{F}^n$

$\Rightarrow T(S_{\mathbb{F}_n})$ is compact in X

X Hausdorff $\Rightarrow T(S_{\mathbb{F}_n})$ is closed

Clearly $0 \notin T(S_{\mathbb{F}_n})$ (T is a linear bijection, $0 \notin S_{\mathbb{F}_n}$)

$\Rightarrow \exists U$ an absolutely convex nbhd of 0 in X s.t. $U \cap T(S_{\mathbb{F}_n}) = \emptyset$

We claim that $U \subset T(U_{\mathbb{F}_n})$ ← open unit ball

Assume $x \in U \setminus T(U_{\mathbb{F}_n})$

$\Rightarrow z := T^{-1}(x)$ satisfies $\|z\|_2 \geq 1$

Then $\frac{z}{\|z\|_2} \in S_{\mathbb{F}_n}$, $T\left(\frac{z}{\|z\|_2}\right) = \frac{1}{\|z\|_2} \cdot T(z) =$

$= \frac{1}{\|z\|_2} x \in U$ (U is balanced)

$\Rightarrow \frac{1}{\|z\|_2} x \in U \cap T(S_{\mathbb{F}_n})$ ↯ a contradiction

$\Rightarrow T^{-1}$ is cts at 0 ($(T^{-1})^{-1}(U_{\mathbb{F}})$ is a nbhd of 0 and the same for all multiples)

$\Rightarrow T^{-1}$ is continuous

So, T is an isomorphism and (S) is proved

(a) By (S) WLOG $X = \mathbb{F}^n$

Let $L : \mathbb{F}^n \to Y$ be linear, Y LCS

Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{F}^n$

Then $L(\lambda_1, \dots, \lambda_n) = \lambda_1 L(e_1) + \dots + \lambda_n L(e_n)$

This is continuous ---- the same argument as in (S) for T works.

$(\lambda_1, \dots, \lambda_n) \mapsto \lambda_j$ is cts

$\lambda \mapsto \lambda L(e_j)$ is cts

take the composition

$(\lambda_1, \dots, \lambda_n) \mapsto \lambda_j L(e_j)$

and sum it up over $j = 1, \dots, n$.)