

SEVERAL EXAMPLES ON EXTREME POINTS

① $X = \mathbb{R}^n$, $K = B_X$, the closed unit ball

• $\text{ext } K = \{\pm e^n, n \in \mathbb{N}\}$ (in the real case):

• $e^n \in \text{ext } K$: $e^n = \frac{1}{2}(x+y)$, $x, y \in B_X$

look at the n -th coordinate: $1 = \frac{1}{2}(x_n + y_n)$

Since $x_n \leq 1$ and $y_n \leq 1$, necessarily $x_n = y_n = 1$.

But since $x, y \in B_X$, necessarily $x = y = e^n$

• $-e^n \in \text{ext } K$ similarly

• $x \in K \setminus \{\pm e^n, n \in \mathbb{N}\}$

either $\|x\| < 1$. Then x is an interior point of a segment in K : $x = 0 \dots$ take $[-e^1, e^1]$

$x \neq 0 \dots$ take $[0, \frac{x}{\|x\|}]$

or $\|x\| = 1$. Then there are $m, n, m < n$

s.t. $x_m \neq 0, x_n \neq 0$

Fix $\varepsilon > 0$, $\varepsilon < \min(|x_m|, |x_n|)$

If $\text{sgn } x_m = \text{sgn } x_n$ define y, z as follows

$$z_k = \begin{cases} x_m + \varepsilon & k = m \\ x_n - \varepsilon & k = n \\ x_k & \text{otherwise} \end{cases}$$

$$y_k = \begin{cases} x_m - \varepsilon & k = m \\ x_n + \varepsilon & k = n \\ x_k & \text{otherwise} \end{cases}$$

Then $y, z \in B_X$, $\|y\| = \|z\| = 1$, $y \neq z$, $x = \frac{y+z}{2}$
 so $x \notin \text{ext } K$

If $\text{sgn } x_m = -\text{sgn } x_n$ define y, z as follows:
 $z_k = \begin{cases} x_m + \varepsilon, & k=m \\ x_n + \varepsilon, & k=n \\ x_k & \text{otherwise} \end{cases}$ $y_k = \begin{cases} x_m - \varepsilon, & k=m \\ x_n - \varepsilon, & k=n \\ x_k & \text{otherwise} \end{cases}$

Then $y, z \in B_X$, $\|y\| = \|z\| = 1$, $y \neq z$, $x = \frac{y+z}{2}$
 so $x \notin \text{ext } K$

• X is a dual space, $\ell_1 = C^*$, then B_X is w^* -compact. Hence, Krein-Milman shows $K = \text{co ext } K$

• Directly: $\text{co ext } K = \{x \in K; \{n; x_n \neq 0\} \text{ is finite}\}$

C: clear from the description of $\text{ext } K$ above

D: $\{n; x_n \neq 0\} = \{n_1 < n_2 < \dots < n_k\}$

Assume $k \geq 1$ (c.o. $x \neq 0$). Then

$$|x_{n_1}| + \dots + |x_{n_k}| \leq 1$$

If the equality holds, then

$$x = |x_{n_1}| \cdot (\text{sgn } x_{n_1}) e^{n_1} + \dots + |x_{n_k}| \cdot (\text{sgn } x_{n_k}) e^{n_k}$$

So $x \in \text{co } \text{eul } k$

If $|x_{n_1}| + \dots + |x_{n_k}| < 1$, let $t = 1 - (|x_{n_1}| + \dots + |x_{n_k}|)$

$$\text{then } x = \frac{t}{2} \cdot (-\text{sgn } x_{n_1} \cdot e^{n_1}) + \left(|x_{n_1}| + \frac{t}{2} \right) \cdot \text{sgn } x_{n_1} e^{n_1}$$

$$+ |x_{n_2}| \text{sgn } x_{n_2} \cdot e^{n_2} + \dots + |x_{n_k}| \text{sgn } x_{n_k} e^{n_k}$$

So, $x \in \text{co } \text{eul } k$

Finally, if $x=0$, then $x = \frac{1}{2} (e^1 + (-e^1)) \in \text{co } \text{eul } k$

• Hence, we have even $k = \overline{\text{co } \text{eul } k} \quad \text{||-11}$

$$(x \in k \Rightarrow x = \lim_{n \rightarrow \infty} (x_{n_1}, \dots, x_{n_k}, 0, 0, \dots)) \quad (\text{in ||-11})$$

$$\text{and } (x_{n_1}, \dots, x_{n_k}, 0, 0, \dots) \in \text{co } \text{eul } k$$

② $X = C(K)^*$, where K is a compact space

$A = P(K)$, the probability measures

• $\text{eul } A = \{ \delta_x, x \in K \}$

$x \in K \Rightarrow \delta_x \in \text{eul } P(K) :$

Assume $\delta_x = \frac{1}{2} (\mu_1 + \mu_2)$, $\mu_1, \mu_2 \in P(K)$

Evaluate at x :

$$1 = \frac{1}{2} (\mu_1(\{x\}) + \mu_2(\{x\}))$$

Since $\mu_1(\{x\}), \mu_2(\{x\}) \in [0, 1]$,

We deduce then $\mu_1(\{x\}) = \mu_2(\{x\}) = 1$

Since μ_1, μ_2 are probabilities, $\mu_1 = \mu_2 = \delta_x$

Thus, indeed, $\delta_x \in \text{ext } \mathcal{P}(K)$

$$\bullet \mu \in \mathcal{P}(K) \setminus \{ \delta_x; x \in K \}$$

Then there are $B_1, B_2 \subset K$ disjoint Borel sets
s.t. $\mu(B_1) > 0, \mu(B_2) > 0$

Let $U = \{ U \subset K \text{ open}; \mu(U) = 0 \}$

Let $G := \cup U$. Then G is open. Moreover,

$\mu(G) = 0 \wedge L \subset G \text{ compact} \Rightarrow \exists F \subset U$ finite
s.t. $L \subset U^c$. Thus $\mu(L) = 0$

μ Radon $\Rightarrow \mu(G) = 0$

$H := K \setminus G \Rightarrow \mu(H) = 1$. μ Borel Dual \Rightarrow
 $\exists x, y \in H, x \neq y$. Let V, W be disjoint open
neighborhoods of x, y .

Then $\mu(V) > 0, \mu(W) > 0$ $\left(\begin{array}{l} \mu \in \mathcal{P}(G) \\ \mu \in \mathcal{P}(G) \end{array} \right)$

Let $\delta, \eta \in (0, 1)$.

Define ν_1, ν_2 by:

$$\nu_1(B) = \mu(B \setminus (B_1 \cup B_2)) + (1 + \delta) \mu(B \cap B_1) + (1 - \eta) \mu(B \cap B_2)$$

$$\nu_2(B) = \mu(B \setminus (B_1 \cup B_2)) + (1 - \delta) \mu(B \cap B_1) + (1 + \eta) \mu(B \cap B_2)$$

Then ν_1, ν_2 are positive measures, $\frac{1}{2}(\nu_1 + \nu_2) = \mu$,

$\nu_1 \neq \nu_2$. By a suitable choice of δ, η

we get $\nu_1, \nu_2 \in \mathcal{P}(K)$:

$$\delta \mu(B_1) = \eta \mu(B_2)$$

and this may be achieved.

Thus $\mu \notin \text{ext } \mathcal{P}(K)$

• X is a dual space, $\mathcal{P}(K)$ is w^* -compact

$$\left(\mathcal{P}(K) = \left\{ \mu \in \mathcal{B}_{\mathcal{C}(K)^*} ; \mu(1) = 1 \right\} \right)$$

So, by Krein-Milman $\mathcal{P}(K) = \overline{\text{co ext } \mathcal{P}(K)}^{w^*}$

• There is also a norm-topology:

$\text{co ext } P(K) =$ finitely supported probabilities

[clear from the above representation]

$\text{co ext } P(K)$ ^(1.1) = countably supported probabilities

$$\mathcal{D}: \mu = \sum_{n=1}^{\infty} t_n \delta_{x_n} \Rightarrow \mu = \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N t_n \delta_{x_n} + \left(\sum_{n=N+1}^{\infty} t_n \right) \delta_{x_{N+1}} \right)$$

$t_n \geq 0, \sum t_n = 1$

the limit in norm:

\subset : Countably supported probabilities from a norm-closed set,

because it is $P(K)$ of countably supported measures

and countably supported measures form a subspace isometric to $\ell_1(K)$,

hence norm-closed.

③ $X = \ell_{\infty}, K = B_X$ (the real version)

• let $K = \{ (x_n) ; t_n : x_n = 1 \text{ or } x_n = -1 \}$

$$D: x = \frac{1}{2}(y+z), \quad y, z \in k$$

$$\Rightarrow \forall n: x_n = \frac{1}{2}(y_n + z_n)$$

$$\text{So } y_n, z_n \in [-1, 1] \text{ and } x_n = 1 \text{ or } x_n = -1,$$

$$\text{clearly } y_n = z_n = x_n.$$

$$\text{So } y = z = x, \text{ hence } x \in \text{ext } k$$

$$C: \text{ Assume } \exists n: x_n \neq 1 \text{ and } x_n \neq -1, \text{ so } x_n \in (-1, 1)$$

$$\text{Find } \varepsilon > 0 \text{ s.t. } [x_n - \varepsilon, x_n + \varepsilon] \subset (-1, 1)$$

let y, z be defined by

$$y_k = \begin{cases} x_n + \varepsilon, & k=n \\ x_k, & \text{otherwise} \end{cases} \quad z_k = \begin{cases} x_n - \varepsilon, & k=n \\ x_k, & \text{otherwise} \end{cases}$$

$$\Rightarrow y, z \in k, \quad y \neq z, \quad \frac{1}{2}(y+z) = x, \text{ so } x \notin \text{ext } k$$

• X is a dual space, $k_\infty = (k_1)^*$, k is w^* -compact.

This Krein-Milman $\Rightarrow k = \overline{\text{co ext } k}^{w^*}$

• Norm topology: Fixed, compute $\text{co ext } k$

$\text{co alk} = \{x \in K; \{x_n, n \in \mathbb{N}\} \text{ is finite}\}$

(i.e., sequences which have only finitely many values)

$$C: x = t_1 y^1 + \dots + t_k y^k, \quad y^1, \dots, y^k \in \text{co alk}$$

$$\text{Then } \forall n: x_n \in \{\pm t_1 \pm t_2 \pm \dots \pm t_k\}$$

where all combinations of signs are allowed

This shows that we have at most 2^k values.

\supset : Assume that $x \in K$ attains finitely many values
 $x = (x_n)$

$$\text{and } \{x_n, n \in \mathbb{N}\} = \{a_1 < a_2 < \dots < a_k\}$$

$$\text{Let } A_j = \{n \in \mathbb{N}; x_n = a_j\}$$

We know that $a_j \in [-1, 1]$,

$$\text{so } a_j = (1 - t_j) \cdot (-1) + t_j \cdot 1,$$

$$\text{where } t_j = \frac{1 + a_j}{2}$$

$$\text{Then } 0 \leq t_1 < t_2 < \dots < t_k \leq 1$$

$$\text{Set } s_1 = t_1, s_2 = t_2 - t_1, \dots, s_k = t_k - t_{k-1}, s_{k+1} = 1 - t_k$$

Let $s_1, \dots, s_{k+1} \geq 0$, $s_1 + \dots + s_{k+1} = 1$

Define $y^1, \dots, y^{k+1} \in \mathbb{R}^n$ as follows:

$$y_n^1 = 1, n \in \mathbb{N}$$

$$y_n^2 = \begin{cases} -1, & n \in A_1 \\ 1, & \text{otherwise} \end{cases}$$

$$y_n^3 = \begin{cases} -1, & n \in A_1 \cup A_2 \\ 1, & \text{otherwise} \end{cases}$$

\vdots

$$y_n^{k+1} = -1, n \in A_1 \cup \dots \cup A_k = \mathbb{N}$$

$$\text{Then } x = s_1 y^1 + s_2 y^2 + \dots + s_{k+1} y^{k+1}$$

$$\forall n \in \mathbb{N} \dots \exists ! j : n \in A_j$$

$$s_1 y_n^1 + \dots + s_j y_n^j + s_{j+1} y_n^{j+1} + \dots + s_{k+1} y_n^{k+1}$$

$$= s_1 + \dots + s_j - s_{j+1} - \dots - s_{k+1}$$

$$= t_j - (1 - t_j) = a_j = x_n$$

- $(\mathbb{C} \text{ or } \mathbb{R})^{\mathbb{N}}$ $\equiv \mathbb{K}$: Any $x \in \mathbb{K}$ can be approximated by elements attaining finitely many values

$$x \in K, \quad m \in \mathbb{N}$$

$$\text{define } A_j^m = \left\{ n \in \mathbb{N}; x_n \in \left[-1 + \frac{j}{m}, -1 + \frac{j+1}{m} \right) \right\}$$

$$j = 0, 1, \dots, 2m$$

$$\text{define } y_n = -1 + \frac{j}{m}, \quad n \in A_j^m$$

$$\text{Then } y \in \text{co ext } K \text{ and } \|y - x\| \leq \frac{1}{m}$$

$$m \in \mathbb{N} \text{ arbitrary} \Rightarrow x \in \overline{\text{co ext } K}^{\|\cdot\|}$$

• Alternative proof that $\text{co ext } K = \text{elements attaining only finitely many values, namely of } a^{\text{th}}$.

Define a_j^- and A_j^- as above.

Note that $a = (a_1, \dots, a_k) \in B_Z$, where

$$Z = (\mathbb{R}^k, \|\cdot\|_\infty) (= \ell_\infty^k).$$

By Minkowski-Carathéodory theorem

$$B_Z = \text{co}(\text{ext } B_Z), \quad \text{more precisely}$$

$$a = \varepsilon_1 b^1 + \dots + \varepsilon_{k+1} b^{k+1}, \quad \text{where } b^1, \dots, b^{k+1} \in \text{ext } B_Z$$

define $y_n^j = b_m^j$ if $n \in A_m, j=1, \dots, k+1$

Then $y_n^j \in \text{ext } K, x = t_1 y^1 + \dots + t_{k+1} y^{k+1}$