Proof of Proposition IX.4

X - HLCS, $\dim X = n$

$K \subset X$ compact convex $\Rightarrow \forall x \in K$: x is a convex combination of points $x_1, \dots, x_k \in X$ with properties:

- $x_1, \dots, x_k \in \operatorname{ext} K$
- $x_1, \dots, x_k$ are affinely independent
- $k \leq n+1$

Recall: $x_1, \dots, x_k$ are affinely independent $\Leftrightarrow$ $x_2 - x_1, \dots, x_k - x_1$ are linearly independent.

Proof ① $n = 0 \Rightarrow K = \{0\}$

$n = 1 \Rightarrow K = [a,b] \subset \mathbb{R}$ or $K = \{a\} \subset \mathbb{R}$ } clear

② Suppose $n \geq 1$ and that it is true if $\dim X \leq n$.

Suppose $\dim X = n+1$

- WLOG $0 \in K$ (replace K by $K - x$ for some $x \in K$)
- WLOG $\operatorname{span} K = X$ (otherwise replace X by $\operatorname{span} K$ and apply the induction hypothesis for $\dim(\operatorname{span} K) \leq n$)
- Then K has nonempty interior

Let $x_1, \dots, x_{n+1} \in K$ be linearly independent, hence a basis of X

Then $T : \mathbb{R}^{n+1} \to X$ defined by

$T(t_1, \dots, t_{n+1}) = \sum_{j=1}^{n+1} t_j \cdot x_j$ is an isomorphism of $\mathbb{R}^{n+1}$ onto X

Then we deduce that $\operatorname{co}\{0, x_1, \dots, x_{n+1}\}$ has nonempty interior, as in $\mathbb{R}^{n+1}$

$\operatorname{co}\{0, e^1, \dots, e^{n+1}\}$ has nonempty interior.

So, $\operatorname{Int} K \neq \emptyset$

- Let $x \in K$. If $x \in \partial K$, then $x \notin \operatorname{Int} K$,
so $\exists f \in X^* \setminus \{0\}$ $f(x) \geq \sup f(\operatorname{Int} K) = \sup f(K)$
(by Thm V.35 (a)).

Int K is dense in K (see Lemma V.7)

So, $F = \{ y \in K,\ f(y) = f(x) \}$
is a closed face
We can use induction hypothesis
on F (its dimension is at most n) [and Lemma IX.2(a, b)]

- $x \in \operatorname{Int} K$. Fix $y \in \operatorname{ext} K$ arbitrary. Consider the
line $y + t(x-y)$, $t \in \mathbb{R}$. This line intersects K in a segment,
one endpoint is y. Denote the second one by z. Then
$z \in \partial K$. Use the previous case to deduce that
$z = t_1 a_1 + \dots + t_k a_k$, $a_1, \dots, a_k \in \operatorname{ext} K$ (in fact,
they are in ext F, where F is the face constructed
in the previous case), $k \leq n$, $a_1, \dots, a_k$
affinely independent.

Since $x = (1-s) y + s z$ for some $s \in (0,1)$, we have
$x = (1-s) y + s t_1 a_1 + \dots + s t_k a_k$

It remains to observe that $y, a_1, \dots, a_k$ are affinely independent.

[Let $t_0 (y - a_1) + t_2 (a_2 - a_1) + \dots + t_k (a_k - a_1) = 0$
Apply the functional f constructed in the first case
then $f(a_1) = f(a_2) = \dots = f(a_k) > f(y)$
Thus $t_0 (f(y) - f(a_1)) = 0$, so $t_0 = 0$

Since $a_1, \dots, a_k$ are aff. independent, we see
$a_2 = \dots = a_k = 0$.]