(a) $L^p(\mu; X)$ is a Banach space

- it is a normed linear space by the definition and Minkowski inequality

completeness:

- $p = \infty$: Let $(f_n)_{n=1}^{\infty}$ be a $\| \|_\infty$ - Cauchy

Then $\forall k \in \mathbb{N} \; \exists n_0 = n_0(k) \; \forall m, n \geq n_0$:
$$\| f_n - f_m \|_\infty < \frac{1}{k}$$

So, $N_{m,n,k} = \{ \omega ; \; \| f_n(\omega) - f_m(\omega) \| \geq \frac{1}{k} \}$

has measure 0 whenever $m, n \geq n_0(k)$

$N := \bigcup_{k \in \mathbb{N}} \bigcup_{m,n \geq n_0(k)} N_{m,n,k} \Rightarrow N$ has measure 0

and (f_n) is uniformly Cauchy on $\Omega \setminus N$.

Since X is complete, it follows that f_n is pointwise convergent on $\Omega \setminus N$. Being moreover uniformly Cauchy, it is uniformly convergent

$f(\omega) := \lim f_n(\omega)$, $\omega \in \Omega \setminus N$

$\varepsilon > 0 \Rightarrow \exists n_0 \; \forall m, n \geq n_0 \; \| f_m - f_n \|_\infty < \varepsilon$

$n \geq n_0$. $\forall \omega \in \Omega \setminus N \; \forall m \geq n_0 \; \| f_n(\omega) - f_m(\omega) \| < \varepsilon$

$\Downarrow$

$\| f_n(\omega) - f(\omega) \|$

$\Rightarrow \| f_n(\omega) - f(\omega) \| \leq \varepsilon$

*) known from measure theory:

- $h_n \to h$ in $L^p(\mu) \Rightarrow h_n \to h$ in measure (i.e. $\forall \varepsilon > 0 \;\; \mu\,[|h_n - h| > \varepsilon] \to 0$)
- $h_n \to h$ in measure $\Rightarrow \exists (h_{n_k}) : h_{n_k} \to h$ a.e.

• $p = \infty$ (f.e. $p \in [1,\infty)$)

Suppose $(f_n) \subset \mathcal{L}^P(\mu; X)$, $\sum_{n=1}^{\infty} \|f_n\|_p < \infty$

Define $g_n(\omega) := \|f_n(\omega)\|$, $\omega \in \Omega$

By the definition $g_n \in \mathcal{L}^P(\mu)$ & $\|g_n\|_p = \|f_n\|_p$

So, $\sum_{n=1}^{\infty} \|g_n\|_p < \infty$. Since $\mathcal{L}^P(\mu)$ is complete,

we get that $\sum_{n=1}^{\infty} g_n$ converges in $\mathcal{L}^P(\mu)$.

$g := \sum_{n=1}^{\infty} g_n \in \mathcal{L}^P(\mu)$ (convergence in $\mathcal{L}^P(\mu)$)

Further, there is a subsequence of the sequence of partial sums converging a.e.*) . But $g_n \geq 0$, so $g(\omega) = \sum_{n=1}^{\infty} g_n(\omega)$ a.e.

Hence, for almost all $\omega \in \Omega$ we have

$$\sum_{n=1}^{\infty} g_n(\omega) < \infty \Rightarrow \sum_{n=1}^{\infty} \|f_n(\omega)\| < \infty \Rightarrow \underbrace{\sum_{n=1}^{\infty} f_n(\omega)}_{\text{converges in } X}$$

Set $f(\omega) = \sum_{n=1}^{\infty} f_n(\omega)$ $\Rightarrow$ f is defined a.e.

Moreover, f is strongly μ-measurable (Lemma 4)

$$\|f(\omega)\| \leq \sum_{n=1}^{\infty} \|f_n(\omega)\| = g(\omega) \quad \text{a.e.} \Rightarrow f \in \mathcal{L}^P(\mu; X)$$

Finally, $\|f(\omega) - \sum_{k=1}^{n} f_k(\omega)\| = \|\sum_{k>n} f_k(\omega)\| \leq \sum_{k>n} \|f_k(\omega)\|$ a.e.

$$\Rightarrow \|f - \sum_{k=1}^{n} f_k(\omega)\|_p \leq \|\omega \mapsto \sum_{k>n} \|f_k(\omega)\|\,\|_p \leq$$

$$\leq \sum_{k>n} \|\underbrace{\omega \to \|f_k(\omega)\|}_{g_k}\|_p = \sum_{k>n} \|f_k\|_p \to 0$$

So, $\sum_{n=1}^{\infty} f_n = f$ in $L^p(\mu; X)$, this completes the proof of completeness.

(b) $L^1(\mu; X)$ = Bochner-integrable functions

[By definitions and Theorem 8]

(c) X Hilbertspace $\Rightarrow$ $L^2(\mu; X)$ is a Hilbertspace

[$\langle f, g \rangle := \int_\Omega \langle f(\omega), g(\omega) \rangle \, d\mu(\omega)$

- $\omega \mapsto \langle f(\omega), g(\omega) \rangle$ is measurable

  [$f(\omega) =$ a.e. $-\lim u_n(\omega)$
  $g(\omega) =$ a.e. $-\lim v_n(\omega)$ $\quad u_n, v_n$ simple measurable

  $\omega \mapsto \langle u_n(\omega), v_n(\omega) \rangle$ is simple measurable
  $\langle f(\omega), g(\omega) \rangle = \lim \langle u_n(\omega), v_n(\omega) \rangle$ a.e.]]

- $\omega \mapsto \langle f(\omega), g(\omega) \rangle$ is integrable

  [$\int_\Omega |\langle f(\omega), g(\omega) \rangle| \, d\mu(\omega) \leq \int_\Omega \|f(\omega)\| \cdot \|g(\omega)\| \, d\mu(\omega)$

  $\leq \left(\int_\Omega \|f(\omega)\|^2 d\mu(\omega) \right)^{1/2} \left(\int_\Omega \|g(\omega)\|^2 d\mu(\omega) \right)^{1/2} = \|f\|_2 \|g\|_2$]]

Hence, $\langle f, g \rangle$ is well-defined. Clearly it is linear in f and $\overline{\langle f, g \rangle} = \langle g, f \rangle$.

Finally, $\langle f, f \rangle = \int_\Omega \langle f(\omega), f(\omega) \rangle d\mu(\omega) = \int_\Omega \|f(\omega)\|^2 d\mu(\omega)$

$= \|f\|_2^2$. So, it is an inner product generating the norm.]