Proposition VIII.7

(a) Integrable simple functions form a vector space and $f \mapsto \int_\Omega f \, d\mu$ is linear.

$\ulcorner$ f, g simple integrable, $\alpha, \beta \in \mathbb{F}$

$$f = \sum_{j=1}^{k} x_j \chi_{E_j} \quad x_1, \dots, x_k \in X, \; E_1, \dots, E_k \in \Sigma \text{ pw disjoint}, \; \bigcup E_j = \Omega$$

$$\forall j = 1 \dots k : \mu(E_j) < \infty \text{ or } x_j = 0$$

$$g = \sum_{m=1}^{\ell} y_m \chi_{A_m} \quad y_1, \dots, y_\ell \in X, \; A_1, \dots, A_\ell \in \Sigma \text{ pw disjoint}, \; \bigcup A_m = \Omega$$

$$\forall m = 1 \dots \ell : \mu(A_m) < \infty \text{ or } y_m = 0$$

$$\alpha f + \beta g = \sum_{j=1}^{k} \sum_{m=1}^{\ell} (\alpha x_j + \beta y_m) \chi_{E_j \cap A_m}$$

$E_j \cap A_m \in \Sigma$ (pw disjoint, covering Ω)

If $\mu(E_j \cap A_m) = \infty$, then both $\mu(E_j) = \infty$ and $\mu(A_m) = \infty$, hence $x_j = y_m = 0 \Rightarrow \alpha x_j + \beta y_m = 0$

So, $\alpha f + \beta g$ is integrable

Moreover,

$$\int_\Omega (\alpha f + \beta g) = \sum_{j=1}^{k} \sum_{m=1}^{\ell} (\alpha x_j + \beta y_m) \mu(E_j \cap A_m) =$$

$$= \sum_{j=1}^{k} \sum_{m=1}^{\ell} \alpha x_j \mu(E_j \cap A_m) + \sum_{j=1}^{k} \sum_{m=1}^{\ell} \beta y_m \mu(E_j \cap A_m) =$$

$$= \alpha \sum_{j=1}^{k} x_j \Big(\sum_{m=1}^{\ell} \mu(E_j \cap A_m) \Big) + \beta \sum_{m=1}^{\ell} y_m \Big(\sum_{j=1}^{k} \mu(E_j \cap A_m) \Big) =$$

$$= \alpha \sum_{j=1}^{k} x_j \mu(E_j) + \beta \sum_{m=1}^{\ell} y_m \mu(A_m) = \alpha \int_\Omega f + \beta \int_\Omega g$$

$\ulcorner$ Any ∞ appearing in the computation is multiplied by 0 $\lrcorner$

$\lrcorner$

(b) Let f be a simple measurable function.
Then f is integrable $\Leftrightarrow$ $\omega \mapsto \|f(\omega)\|$ is integrable.
In this case $\quad \left\| \int_\Omega f \, d\mu \right\| \le \int_\Omega \|f\| \, d\mu$

$\ulcorner$ $f = \sum_{j=1}^{k} x_j \chi_{E_j}$, $\quad E_j \in \Sigma$ pw disjoint

Then $\|f(\omega)\| = \sum_{j=1}^{k} \|x_j\| \chi_{E_j}(\omega)$, $\quad \omega \in \Omega$

Hence it is a simple measurable function.
Moreover, since $x_j = 0 \Leftrightarrow \|x_j\| = 0$, the integrability of f and $\omega \mapsto \|f(\omega)\|$ is equivalent.

$\int_\Omega f \, d\mu = \sum_{j=1}^{k} x_j \mu(E_j)$, $\quad \int_\Omega \|f(\omega)\| \, d\mu(\omega) = \sum_{j=1}^{k} \|x_j\| \mu(E_j)$,

hence $\quad \left\| \int_\Omega f \, d\mu \right\| \le \int_\Omega \|f(\omega)\| \, d\mu(\omega)$ by the triangle inequality. $\lrcorner$

(c) The limit defining the Bochner integral exists and does not depend on the choice of (f_n)

$\ulcorner$ Existence: Let (f_n) be a sequence of simple integrable measurable functions s.t. $\int_\Omega \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) \xrightarrow{n\to\infty} 0$

$\varepsilon > 0 \Rightarrow \exists n_0 \;\; \forall n \ge n_0 : \int_\Omega \|f_n(\omega) - f(\omega)\| \, d\mu(\omega) < \frac{\varepsilon}{2}$

Then for $m, n \ge n_0$

$$\int_\Omega \|f_m(\omega) - f_n(\omega)\| \, d\mu(\omega) \le \int_\Omega \left(\|f_m(\omega) - f(\omega)\| + \|f(\omega) - f_n(\omega)\| \right) d\mu(\omega)$$

$$= \int_\Omega \|f_m(\omega) - f(\omega)\| \, d\mu(\omega) + \int_\Omega \|f(\omega) - f_n(\omega)\| \, d\mu(\omega) < \varepsilon$$

Thus, $\omega \mapsto \|f_m(\omega) - f_n(\omega)\|$ is an integrable simple function, so, by (b) $f_m - f_n$ is integrable and

$$\left\| \int_\Omega (f_m - f_n)\, d\mu \right\| \le \int_\Omega \|f_m(\omega) - f_n(\omega)\|\, d\mu(\omega) < \varepsilon.$$

By (a) we know

$$\left\| \int_\Omega f_m\, d\mu - \int_\Omega f_n\, d\mu \right\| = \left\| \int_\Omega (f_m - f_n)\, d\mu \right\| < \varepsilon$$

So, we have proved that the sequence $\left(\int_\Omega f_n\, d\mu \right)_{n=1}^{\infty}$ is a Cauchy sequence in X. So, it converges.

The integral does not depend on the choice of (f_n): If (f_n) and (g_n) are two sequences of simple integrable functions s.t. $\int \|f(\omega) - f_n(\omega)\|\, d\mu(\omega) \to 0$ and $\int \|g(\omega) - g_n(\omega)\|\, d\mu(\omega) \to 0$, then the sequence $f_1, g_1, f_2, g_2, f_3, g_3, \dots$ satisfies the same property. $\lrcorner$

(d) Bochner integrable functions form a vector space and $f \mapsto \int_\Omega f\, d\mu$ is a linear mapping.

$\ulcorner$ f, g Bochner integrable, $\alpha, \beta \in \mathbb{F}$
(f_n) ... a sequence of simple integrable functions for f
(g_n) ——— " ——— g

then $\alpha f_n + \beta g_n$ are simple integrable functions (by (a)),

$$\int_\Omega \|\alpha f(\omega) + \beta g(\omega) - (\alpha f_n(\omega) + \beta g_n(\omega))\|\, d\mu(\omega) \le |\alpha| \int_\Omega \|f(\omega) - f_n(\omega)\|\, d\mu(\omega)$$

$$+ |\beta| \int_\Omega \|g(\omega) - g_n(\omega)\|\, d\mu(\omega) \to 0$$

So, $\alpha f + \beta g$ is B-integrable, and

$$(B)\int_\Omega (\alpha f+\beta g)\,d\mu = \lim_{n\to\infty} \int_\Omega (\alpha f_n + \beta g_n)\,d\mu \overset{(a)}{=} \lim_{n\to\infty} \alpha\int_\Omega f_n d\mu + \beta\int_\Omega g_n\,d\mu$$

$$= \alpha \cdot \lim_{n\to\infty} \int_\Omega f_n d\mu + \beta \cdot \lim_{n\to\infty} \int_\Omega g_n d\mu = \alpha \cdot (B)\int_\Omega f d\mu + \beta (B)\int_\Omega g\,d\mu$$

topology of a Banach space is linear.

(e) f Bochner integrable $\Rightarrow$ $\omega \mapsto \|f(\omega)\|$ is integrable and

$$\|(B)\int_\Omega f\,d\mu\| \leq \int_\Omega \|f(\omega)\|\,d\mu(\omega)$$

Let f be Bochner integrable. Then f is strongly μ-measurable hence $\omega \mapsto \|f(\omega)\|$ is measurable by Prop. 1

Moreover, for some $n \in \mathbb{N}$ $\int_\Omega \|f_n(\omega) - f(\omega)\|\,d\mu(\omega) < \infty$,

Since $\|f(\omega)\| \leq \|f(\omega) - f_n(\omega)\| + \|f_n(\omega)\|$, we conclude that $\omega \mapsto \|f(\omega)\|$ is integrable

The estimate:

$$\|(B)\int_\Omega f d\mu\| = \lim_{n\to\infty} \|\int_\Omega f_n d\mu\| \overset{(b)}{\leq} \liminf_{n\to\infty} \int_\Omega \|f_n(\omega)\|\,d\mu(\omega)$$

definition + continuity of the norm

$$\leq \liminf_{n\to\infty} \int_\Omega (\|f_n(\omega) - f(\omega)\| + \|f(\omega)\|)\,d\mu(\omega) =$$

$$= \underbrace{\left(\liminf_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\|\,d\mu(\omega)\right)}_{=0} + \int_\Omega \|f(\omega)\|\,d\mu(\omega) = \int_\Omega \|f(\omega)\|\,d\mu(\omega)$$

(f) f Bochner integrable, $E \in \Sigma \Rightarrow \chi_E \cdot f$ is Bochner integrable

Γ (f_n) the defining sequence for f. The $\chi_E \cdot f_n$ are simple integrable functions and

$$\int_\Omega \| \chi_E(\omega) f(\omega) - \chi_E(\omega) f_n(\omega) \| \, d\mu(\omega) = \int_E \| f(\omega) - f_n(\omega) \| \, d\mu(\omega) \leq$$

$$\leq \int_\Omega \| f(\omega) - f_n(\omega) \| \, d\mu(\omega) \to 0 . \lrcorner$$

Thm VIII.8 f strongly measurable $\Rightarrow$ (f Bochner-integrable $\Leftrightarrow \int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$)

Pf: $\Rightarrow$ By (e) of the previous proposition

$\Leftarrow$ Let f be strongly measurable & $\int_\Omega \|f(\omega)\| d\mu(\omega) < \infty$

$\exists (h_n)$, simple measurable functions s.t. $h_n \to f$ a.e.

Define

$$g_n(\omega) = \begin{cases} h_n(\omega) & \text{if } \|h_n(\omega)\| < 2\|f(\omega)\| \\ 0 & \text{otherwise} \end{cases}$$

Then g_n is a simple function ($g_n(\Omega) \subset h_n(\Omega) \cup \{0\}$)

g_n measurable $g_n^{-1}(\{x\}) = h_n^{-1}(\{x\}) \cap \{\omega ; \|h_n(\omega)\| < 2\|f(\omega)\|\}$
$\in \Sigma$ for $x \in h_n(\Omega)$

g_n integrable, as $\|g_n(\omega)\| \leq 2\|f(\omega)\|$, $\omega \in \Omega$
$\Rightarrow \int \|g_n(\omega)\| d\mu(\omega) < \infty$ and use (b) of the previous Prop.

$g_n \to f$ a.e. Γ Fix $\omega \in \Omega$ s.t. $h_n(\omega) \to f(\omega)$

If $f(\omega) = 0$, then $g_n(\omega) = 0$ for each $n \in \mathbb{N}$

If $f(\omega) \neq 0$, then $\exists n_0\ \forall n \geq n_0\ \|h_n(\omega)\| < 2\|f(\omega)\|$

so, for $n \geq n_0$ $g_n(\omega) = h_n(\omega) \to f(\omega)$ $\lrcorner$

Hence $\|g_n(\omega) - f(\omega)\| \to 0$ a.e. Moreover,

$\|g_n(\omega) - f(\omega)\| \leq \|g_n(\omega)\| + \|f(\omega)\| \leq 2\|f(\omega)\| + \|f(\omega)\| = 3\|f(\omega)\|$

So, by Lebesgue dom. conv. thm $\int_\Omega \|g_n(\omega) - f(\omega)\| d\mu(\omega) \to 0$.