Def: $U \in \mathcal{S}'(\mathbb{R}^d)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ $\Rightarrow$

$$U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x-y))$$

Remark: $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow U*\varphi$ coincides with the notion from Section VII.4

Theorem VII.28

(a) $U * \varphi \in C^\infty(\mathbb{R}^d)$, $D^\alpha(U*\varphi) = D^\alpha U * \varphi = U * D^\alpha \varphi$

[• $\varphi \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \tau_x \check{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ (cf. Prop. VII.22 and its proof), so $U*\varphi$ is well defined

• $U*\varphi$ is cts by Lemma VII.26 (a)

[$x_n \to x \Rightarrow \tau_{x_n}\check{\varphi} \to \tau_x \check{\varphi} \Rightarrow U*\varphi(x_n) \to U*\varphi(x)$]]

• $\frac{\partial}{\partial x_j} U*\varphi(x) = \lim_{t\to 0} \frac{U*\varphi(x+te_j) - U*\varphi(x)}{t} = \lim_{t\to 0} U\left(\frac{\tau_{x+te_j}\check{\varphi} - \tau_x\check{\varphi}}{t}\right)$

$$= \lim_{t\to 0} U\Bigg(\underbrace{\frac{\tau_{te_j}\tau_x\check{\varphi} - \tau_x\check{\varphi}}{t}}_{\substack{\xrightarrow[t\to 0]{} \partial_{-e_j}\tau_x\check{\varphi} = -\partial_{e_j}(y\mapsto \varphi(x-y)) = D^{e_j}\varphi(x-y) \\ \text{by Lem VII.26 (b)}}}\Bigg) = U(y \mapsto D^{e_j}\varphi(x-y)) = \left(U * \frac{\partial\varphi}{\partial x_j}\right)(x)$$

So, $\frac{\partial}{\partial x_j} U*\varphi$ is cts (apply the previous step to $\frac{\partial\varphi}{\partial x_j}$ in place of φ)

Moreover $\left(\frac{\partial}{\partial x_j}U\right)*\varphi = \left(\frac{\partial}{\partial x_j}U\right)(y\mapsto\varphi(x-y)) = -U\left(y\mapsto -\frac{\partial\varphi}{\partial x_j}(x-y)\right) =$

So, $U*\varphi \in C^1$ and the formulas hold for $|\alpha| = 1$.

• By induction on $|\alpha|$. Assume it holds for $|\alpha| \le N$. Let $|\alpha| = N+1$

Then $\alpha = \beta + e_j$, $|\beta| = N$, $j \in \{1,\dots,d\}$

$$D^\alpha(U*\varphi) = D^{e_j}D^\beta(U*\varphi) \underset{\text{induction hypothesis}}{=} D^{e_j}(U * D^\beta\varphi) \underset{\text{case } |\alpha|=1}{=} U * D^{e_j}D^\beta\varphi = U * D^\alpha\varphi$$

cts by the first step

$$= D^{e_j}(D^\beta U * \varphi) = D^{e_j}D^\beta U * \varphi = D^\alpha U * \varphi$$

]

(b) $\Lambda_{U*\varphi}$ is a tempered distribution

$\ulcorner$ Prop. VII.18 $\Rightarrow \exists N, C : |U(\psi)| \leq C\, p_N(\psi),\ \psi \in \mathcal{S}(\mathbb{R}^d)$

$|U*\varphi(x)| = |U(\tau_x \check{\varphi})| \leq C \cdot p_N(\tau_x \check{\varphi})$

$|\alpha| \leq N \ldots\ |(1+\|y\|^2)^N D^\alpha (y \mapsto \varphi(x-y))| = (1+\|y\|^2)^N |D^\alpha \varphi(x-y)| \leq (1+\|y\|^2)^N \cdot \frac{p_N(\varphi)}{(1+\|x-y\|^2)^N}$

$\ulcorner \frac{1+\|y\|^2}{1+\|x-y\|^2} = \frac{1+\|y-x+x\|^2}{1+\|x-y\|^2} = \frac{1+\|y-x\|^2+2\langle y-x, x\rangle+\|x\|^2}{1+\|x-y\|^2} \leq$

$\frac{1+\|y-x\|^2+2\|y-x\|\|x\|+\|x\|^2}{1+\|x-y\|^2} = 1 + \underbrace{\frac{2\|y-x\|\|x\|}{1+\|y-x\|^2}}_{\leq 1} + \frac{\|x\|^2}{\underbrace{1+\|y-x\|^2}_{\geq 1}}$

$\leq 1+\|x\|+\|x\|^2 \lrcorner$

So, $p_N(\tau_x\check{\varphi}) \leq (1+\|x\|+\|x\|^2)^N p_N(\varphi)$,

hence $|U*\varphi(x)| \leq C \cdot (1+\|x\|+\|x\|^2)^N p_N(\varphi) \leq C \cdot (2+2\|x\|^2)^N p_N(\varphi)$

$\Rightarrow \Lambda_{U*\varphi}$ is a tempered distribution by Prop. VII.20 (c). $\lrcorner$

(c) $f \in L^p(\mathbb{R}^d)$ for some $p \in [1,\infty] \Rightarrow \Lambda_f * \varphi = f*\varphi$

$\ulcorner \Lambda_f * \varphi(x) = \Lambda_f(y \mapsto \varphi(x-y)) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)\,dm_d(y) = f*\varphi(x)$ ✓

(d) $\widehat{\Lambda_{U*\varphi}} = \hat{\varphi}\cdot\hat{U}$, $\widehat{\varphi \cdot U} = \Lambda_{\hat{\varphi}*\hat{U}}$

$\ulcorner \widehat{\Lambda_{U*\varphi}}(\psi) = \Lambda_{U*\varphi}(\hat{\psi}) = \int_{\mathbb{R}^d} (U*\varphi)(x)\,\hat{\psi}(x)\,dm_d(x) = \int_{\mathbb{R}^d} U(y \mapsto \varphi(x-y))\,\hat{\psi}(x)\,dm_d(x)$

$= \int_{\mathbb{R}^d} U(y \mapsto \varphi(x-y)\hat{\psi}(x))\,dm_d(x) = U\left(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y)\hat{\psi}(x)\,dm_d(x)\right) = U(\check{\varphi} * \hat{\psi})$

Thm IV.14(b)

$= U(\hat{\hat{\varphi}} * \hat{\psi}) = U(\widehat{\hat{\varphi}\cdot\psi}) = \hat{U}(\hat{\varphi}\cdot\psi) = \hat{\varphi}\cdot\hat{U}(\psi)$

Prop. IV.16

$$\Lambda_{\hat{\varphi} * \hat{U}} = \Lambda_{\widehat{\widehat{\widehat{\widehat{\hat{\varphi} * \hat{U}}}}}} = \widehat{\widehat{\widehat{\hat{\varphi} \cdot \hat{U}}}} = \widehat{\widehat{\widehat{\check{\varphi} \cdot \check{U}}}} = \widehat{\widecheck{\check{\varphi}\, \check{U}}} = \widehat{\check{\check{\varphi}} \cdot \check{U}} = \widehat{\varphi \cdot U} \quad \lrcorner$$

(e) $U * (\varphi * \psi) = (U * \varphi) * \psi$

$$\ulcorner \quad U * (\varphi * \psi)(x) = U\left(y \mapsto \varphi * \psi(x-y)\right) = U\left(y \mapsto \int_{\mathbb{R}^d} \varphi(x-y-z)\,\psi(z)\, dm_d(z)\right)$$

$$= \int_{\mathbb{R}^d} U\left(y \mapsto \varphi(x-y-z)\,\psi(z)\right) dm_d(z) = \int_{\mathbb{R}^d} \psi(z)\, U\left(y \mapsto \varphi(x-y-z)\right) dm_d(z)$$

$$= \int_{\mathbb{R}^d} \psi(z)\, U * \varphi\,(x-z)\, dm_d(z) = (U * \varphi) * \psi\,(x) \quad \lrcorner$$