Prop. VII. 20

(a) $\Lambda \in \mathcal{D}'(\mathbb{R}^d)$ with compact support $\Rightarrow \Lambda$ is tempered

$\ulcorner$ Λ has compact support $\overset{\text{Prop. VII. 12 (d)}}{\Longrightarrow}$ $\exists\, C, N: |\Lambda(\varphi)| \le C \cdot \|\varphi\|_N, \ \varphi \in \mathcal{D}(\mathbb{R}^d)$

Note that $\|\varphi\|_N \le p_N(\varphi)$, hence we may apply Prop. VII. 18 (b). $\lrcorner$

(b) $f \in L^p(\mathbb{R}^d)$ for some $p \in [1,\infty] \Rightarrow \Lambda_f$ is tempered and, moreover,

$$\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

$\ulcorner$ Theorem IV. 11 (a) $\Rightarrow \mathcal{S}(\mathbb{R}^d) \subset \bigcap_{p \in [1,\infty]} L^p(\mathbb{R}^d)$

So, fix $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^d)$. Let p' be the dual exponent.
Then $\forall \varphi \in \mathcal{S}(\mathbb{R}^d)$: $\varphi \in L^{p'}(\mathbb{R}^d)$ and hence $f\varphi$ is integrable

and $\left|\int_{\mathbb{R}^d} f\varphi\right| \le \|f\|_{L^p} \|\varphi\|_{L^{p'}}$.

Thus the above formula for Λ_f defines a linear functional on $\mathcal{S}(\mathbb{R}^d)$, extending the regular distribution Λ_f.

Continuity:

$p = 1$: $|\Lambda_f(\varphi)| = \left|\int_{\mathbb{R}^d} f\varphi\right| \le \|f\|_1 \|\varphi\|_\infty = \|f\|_1\, p_0(\varphi)$.

$p > 1$: $\forall n \in \mathbb{N}$ $f \cdot 1_{U(0,n)} \in L^1(\mathbb{R}^d) \Rightarrow \Lambda_{f \cdot U(0,n)} \in \mathcal{S}'(\mathbb{R}^d)$ by the case $p = 1$

For each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^d} f\varphi \cdot 1_{U(0,n)} = \lim_{n\to\infty} \Lambda_{f 1_{U(0,n)}}(\varphi)$$

So, $\Lambda_f \in \mathcal{S}'(\mathbb{R}^d)$ by Thm 19

/ alternative proof:
We use the following fact (see the remark before Theorem IV. 11):

FACT: if $m > \frac{d}{2}$, then the function $x \mapsto \frac{1}{(1+\|x\|^2)^m}$ is integrable on $\mathbb{R}^d$.

Hence: Fix m s.t. $m \cdot p' > \frac{d}{2}$.

Then $x \mapsto \frac{1}{(1+\|x\|^2)^m}$ belongs to $L^{p'}(\mathbb{R}^d)$.

Thus $x \mapsto \frac{f(x)}{(1+\|x\|^2)^m}$ belongs to $L^1(\mathbb{R}^d)$ (Hölder)

Thus $|\Lambda_f(\varphi)| = \left|\int_{\mathbb{R}^d} f\varphi\right| \leq \left|\int_{\mathbb{R}^d} \frac{f(x)}{(1+\|x\|^2)^m} \cdot (1+\|x\|^2)^m \varphi(x)\,dx\right|$

$\leq \left(\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+\|x\|^2)^m}\,dx\right) \cdot p_m(\varphi)$.

(c) f measurable on $\mathbb{R}^d$, $\exists P$ polynomial s.t. $|f| \leq |P|$ on $\mathbb{R}^d$.
Then Λ_f is tempered and $\Lambda_f(\varphi) = \int_{\mathbb{R}^d} f\varphi$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

P polynomial $\Rightarrow$ $P(x) = \sum_{|\alpha| \leq N} c_\alpha x^\alpha$ ($c_\alpha \in \mathbb{F}$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$)

Then $|P(x)| \leq \sum_{|\alpha| \leq N} |c_\alpha| |x^\alpha| \leq C \cdot \sum_{|\alpha| \leq N} |x|^\alpha \leq C \cdot \prod_{j=1}^{d} (1 + |x_j| + \cdots + |x_j|^N)$

$C := \max_{|\alpha| \leq N} |c_\alpha|$

$\leq C \cdot \prod_{j=1}^{d} (1+|x_j|)^N \leq C \cdot \prod_{j=1}^{d} \left(\sqrt{2} \cdot \sqrt{1+x_j^2}\right)^N \leq C (\sqrt{2})^{dN} \cdot (1+\|x\|^2)^{N \cdot \frac{d}{2}}$

So, if $|f| \leq |P|$, then

$$\left|\frac{f(x)}{(1+\|x\|^2)^m}\right| \leq C \cdot (\sqrt{2})^{dN} (1+\|x\|^2)^{N\frac{d}{2} - m}.$$

If m is large enough ($N\frac{d}{2} - m < -\frac{d}{2}$, i.e. $m > (N+1)\frac{d}{2}$)

then $x \mapsto \frac{f(x)}{(1+\|x\|^2)^m}$ belongs to $L^1(\mathbb{R}^d)$, so similarly

as in the proof of (b) we get that Λ_f is well defined
and $|\Lambda_f(\varphi)| \leq \left(\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+\|x\|^2)^m}\,dx\right) \cdot p_m(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

(d) μ a measure (finite - signed or complex) on $\mathbb{R}^d$
$\Rightarrow \Lambda_\mu$ is tempered and $\Lambda_\mu(\varphi) = \int_{\mathbb{R}^d} \varphi \, d\mu$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

$\ulcorner$ μ finite, φ bdd $\Rightarrow$ φ μ-integrable

$$|\Lambda_\mu(\varphi)| = \left|\int_{\mathbb{R}^d} \varphi \, d\mu\right| \le \|\varphi\|_\infty \cdot \|\mu\| = \|\mu\| \cdot p_0(\varphi) \quad \lrcorner$$

OPERATIONS ON TEMPERED DISTRIBUTIONS:

- Any tempered distribution is a distribution. Hence, if $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$, the following distributions may be considered:
  $D^\alpha \Lambda$ for $\alpha \in \mathbb{N}_0^d$, $f \cdot \Lambda$ for $f \in C^\infty(\mathbb{R}^d)$, $\tau_y \Lambda$ for $y \in \mathbb{R}^d$, $\check{\Lambda}$,
  $\varphi * \Lambda$ for $\varphi \in \mathcal{D}(\mathbb{R}^d)$.
- At the moment we know that the results of these operations are distributions, possibly not tempered.
- We collect some cases, when the results are tempered distributions.

<u>Lemma VII.21</u> The following mappings are cts linear mappings $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$:
- $f \mapsto P.f$ if P is a polynomial
- $f \mapsto g.f$ if $g \in \mathcal{S}(\mathbb{R}^d)$
- $f \mapsto D^\alpha f$ if $\alpha \in \mathbb{N}_0^d$.

<u>Proof:</u> It follows from Prop. V.13 that a linear mapping $L: \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is continuous IFF $\forall N \in \mathbb{N}_0\ \exists C > 0\ \exists M \in \mathbb{N}_0 : p_N(L(f)) \le C \cdot p_M(f)$
for $f \in \mathcal{S}(\mathbb{R}^d)$.

Since the mappings are linear, we will use this characterization.

- Derivatives: $\alpha \in \mathbb{N}_0^d$. Consider the mapping $f \mapsto D^\alpha f$

$\ulcorner$ $f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow D^\alpha f \in C^\infty(\mathbb{R}^d)$

Further, fix $N \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^d$ with $|\beta| \le N$.

Then for $x \in \mathbb{R}^d$:

$$|(1+\|x\|^2)^N D^\beta (D^\alpha f)(x)| = |(1+\|x\|^2)^N D^{\beta+\alpha} f(x)| \le$$
$$\le |(1+\|x\|^2)^{N+|\alpha|} D^{\beta+\alpha} f(x)| \le p_{N+|\alpha|}(f).$$

It follows that $D^\alpha f \in \mathcal{S}(\mathbb{R}^d)$ and $p_N(D^\alpha f) \le p_{N+|\alpha|}(f)$. $\lrcorner$

- P polynomial, $f \mapsto Pf$

$\ulcorner$ □ P polynomial, $f \in \mathcal{S}(\mathbb{R}^d) \Rightarrow Pf \in C^\infty(\mathbb{R}^d)$

□ We know that $\forall Q$ polynomial $\exists\ C > 0,\ m \in \mathbb{N}$ s.t.

$$|Q(x)| \le C \cdot (1+\|x\|^2)^m,\ x \in \mathbb{R}^d$$

$\ulcorner$ see the proof of Prop. VII.19 (c) $\lrcorner$

□ Fix $N \in \mathbb{N}_0$. Then $\exists\ C > 0,\ m \in \mathbb{N}$ s.t.

$$\forall \alpha,\ |\alpha| \le N\ \forall x \in \mathbb{R}^d : \quad |D^\alpha P(x)| \le C\,(1+\|x\|^2)^m$$

$\ulcorner$ Apply the previous step to $D^\alpha P$, $|\alpha| \le N$ and take maxima of the respective values of C, m $\lrcorner$

□ Let $|\alpha| \le N$. Let us compute (for $x \in \mathbb{R}^d$):

$$|(1+\|x\|^2)^N D^\alpha (Pf)(x)| = (1+\|x\|^2)^N \Big|\sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta P(x)\, D^{\alpha-\beta} f(x)\Big|$$

$$\le (1+\|x\|^2)^N \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \cdot C\,(1+\|x\|^2)^m\, |D^{\alpha-\beta} f(x)|$$

$$= C \cdot \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (1+\|x\|^2)^{N+m} |D^{\alpha-\beta} f(x)| \le C \cdot \sum_{\beta \le \alpha} \binom{\alpha}{\beta}\, p_{N+m}(f)$$

$$\le C \cdot 2^N\, p_{N+m}(f).$$

It follows that $p_N(P\cdot f) \le C \cdot 2^N \cdot p_{N+m}(f)$.

□ We deduce that $Pf \in \mathcal{S}(\mathbb{R}^d)$ whenever $f \in \mathcal{S}(\mathbb{R}^d)$ and, moreover, $f \mapsto P\cdot f$ is cts $\lrcorner$

- $g \in \mathcal{S}(\mathbb{R}^d)$, $f \mapsto g\cdot f$

$\ulcorner$ □ $g \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \forall \alpha \in \mathbb{N}_0^d :\ D^\alpha g$ is bdd on $\mathbb{R}^d$

□ Fix $N \in \mathbb{N}_0$. Let $C := \max_{|\alpha| \le N} \|D^\alpha g\|_\infty$.

□ Let $|\alpha| \leq N$ and $x \in \mathbb{R}^d$. Then

$$|(1+\|x\|^2)^N D^\alpha (gf)(x)| = (1+\|x\|^2)^N \Big|\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta g(x)\, D^{\alpha-\beta} f(x)\Big| \leq$$

$$\leq (1+\|x\|^2)^N \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \underbrace{|D^\beta g(x)|}_{\leq C} \cdot |D^{\alpha-\beta} f(x)| \leq$$

$$\leq C \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (1+\|x\|^2)^N |D^{\alpha-\beta} f(x)| \leq C \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} P_N(f) \leq C \cdot 2^N \cdot P_N(f).$$

□ Since $fg \in C^\infty(\mathbb{R}^d)$, we deduce that $fg \in \mathcal{S}(\mathbb{R}^d)$ (for $f \in \mathcal{S}(\mathbb{R}^d)$) and $f \mapsto fg$ is cts ┘

Remark: The same method shows that $f \mapsto g \cdot f$ is a cts mapping $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ whenever $g \in C^\infty(\mathbb{R}^d)$ is such that $\forall \alpha\ \exists P_\alpha$ polynomial s.t. $|D^\alpha g| \leq |P_\alpha|$

Prop. VII.22 Let $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$

(a) $\forall \alpha \in \mathbb{N}_0^d$: $D^\alpha \Lambda \in \mathcal{S}'(\mathbb{R}^d)$. Moreover $D^\alpha \Lambda(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$\varphi \in \mathcal{S}(\mathbb{R}^d) \Rightarrow D^\alpha \varphi \in \mathcal{S}(\mathbb{R}^d)$, so $\tilde{\Lambda}(\varphi) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is a well-defined linear functional on $\mathcal{S}(\mathbb{R}^d)$, whose restriction to $\mathcal{D}(\mathbb{R}^d)$ is $D^\alpha \Lambda$. Continuity:

One possibility: Assume $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d)$. By Lemma VII.20 $D^\alpha \varphi_n \to D^\alpha \varphi$ in $\mathcal{S}(\mathbb{R}^d)$. So $\tilde{\Lambda}(\varphi_n) = (-1)^{|\alpha|} \Lambda(D^\alpha \varphi_n) \to (-1)^{|\alpha|} \Lambda(D^\alpha \varphi) = \tilde{\Lambda}(\varphi)$

Other possibility: $|\tilde{\Lambda}(\varphi)| = |\Lambda(D^\alpha \varphi_n)| \leq C \cdot P_N(D^\alpha \varphi) \leq C \cdot P_{N+|\alpha|}(\varphi)$

C, N from Prop. VII.18 ↑ proof of L VII.20

┘

(b) Assume $f \in \mathcal{S}(\mathbb{R}^d)$ or f is a polynomial. Then $f\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ and $f\Lambda(\varphi) = \Lambda(f\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$

⌈ In both cases $f\varphi \in \mathcal{S}(\mathbb{R}^d)$ for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (by Lemma VII.20)
Thus $\tilde{\Lambda}(\varphi) = \Lambda(f\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is a well-defined linear functional on $\mathcal{S}(\mathbb{R}^d)$ which coincides with $f\Lambda$ on $\mathcal{D}(\mathbb{R}^d)$. It remains to prove it is continuous.

<u>One possibility</u>: $\varphi_n \to \varphi$ in $\mathcal{S}(\mathbb{R}^d) \overset{L\,VII.20}{\Longrightarrow} f\varphi_n \to f\varphi$ in $\mathcal{S}(\mathbb{R}^d)$
$\Rightarrow \tilde{\Lambda}(\varphi_n) = \Lambda(f\varphi_n) \to \Lambda(f\varphi) = \tilde{\Lambda}(\varphi)$.

<u>Other possibility</u>: $|\tilde{\Lambda}(\varphi)| = |\Lambda(f\varphi)| \underset{\substack{\exists C>0\, \exists N \\ \text{by Prop. VII.18}}}{\leq} C \cdot p_N(f\varphi) \leq \begin{cases} C \cdot C' \cdot 2^N p_{N+m}(\varphi) & \text{if } f \text{ is a polynomial} \\ C \cdot C' \cdot 2^N p_N(\varphi) & \text{if } f \in \mathcal{S}(\mathbb{R}^d) \end{cases}$

by the proof of Lemma VII.20. ⌋

(c) $y \in \mathbb{R}^d \Rightarrow \tau_y \Lambda \in \mathcal{S}'(\mathbb{R}^d)$. Moreover, $\tau_y \Lambda(\varphi) = \Lambda(\tau_{-y}\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$

⌈ Assume $y \in \mathbb{R}^d$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then $\tau_{-y}\varphi \in \mathcal{S}(\mathbb{R}^d)$

- Clearly $\tau_{-y}\varphi \in C^\infty(\mathbb{R}^d)$ (recall that $\tau_{-y}\varphi(x) = \varphi(x+y)$)
- $(1+\|x\|^2)^N D^\alpha(\tau_{-y}\varphi)(x) = (1+\|x\|^2)^N D^\alpha \varphi(x+y) =$

$$= \frac{(1+\|x\|^2)^N}{(1+\|x+y\|^2)^N} \underbrace{(1+\|x+y\|^2)^N D^\alpha \varphi(x+y)}_{\text{bdd on } \mathbb{R}^d \text{ as } \varphi \in \mathcal{S}(\mathbb{R}^d)}$$

⌈ $\dfrac{1+\|x\|^2}{1+\|x+y\|^2} \leq \dfrac{1+\|x\|^2}{1+(\|x\|-\|y\|)^2} \leq M$

Consider $h(t) = \dfrac{1+t^2}{1+(t-s)^2}$, $t \in [0,\infty)$, where $s = \|y\|$.

Then h is cts on $[0,\infty)$, $h(0) = \dfrac{1}{1+s^2}$, $\lim\limits_{t\to+\infty} h(t) = 1$
$\Rightarrow h$ is bdd on $[0,\infty)$
Let $M = \sup\limits_{t\in[0,\infty)} h(t)$, then ⌋⌋

we deduce $\tau_{-y}\varphi \in \mathcal{S}(\mathbb{R}^d)$ and, moreover

$$p_N(\tau_{-y}\varphi) \leq M^N \cdot p_N(\varphi)$$

- $\tilde{\Lambda}(\varphi) = \Lambda(\tau_{-y}\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is a well-defined linear functional on $\mathcal{S}(\mathbb{R}^d)$, $\tilde{\Lambda}|_{\mathcal{D}(\mathbb{R}^d)} = \tau_y \Lambda$

Moreover, $\tilde{\Lambda}$ is cts;

$$|\tilde{\Lambda}(\varphi)| = |\Lambda(\tau_{-y}\varphi)| \underset{\exists C, N \text{ by Prop. VII.18}}{\leq} C\, p_N(\tau_{-y}\varphi) \leq C \cdot M^N \cdot p_N(\varphi)$$

(d) $\check{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$. Moreover $\check{\Lambda}(\varphi) = \Lambda(\check{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$\varphi \in \mathcal{S}(\mathbb{R}^d) \Rightarrow \check{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}^d$. Clearly $\check{\varphi} \in C^\infty(\mathbb{R}^d)$. $(1+|x|^2)^N D^\alpha \check{\varphi}(x) = (-1)^{|\alpha|}(1+|x|^2)^N D^\alpha\varphi(-x)$.

$\Rightarrow \check{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ and $p_N(\check{\varphi}) = p_N(\varphi)$ for each $N \in \mathbb{N}_0$. Define $\tilde{\Lambda}(\varphi) = \Lambda(\check{\varphi})$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

$\Rightarrow \tilde{\Lambda}$ is a linear functional on $\mathcal{S}(\mathbb{R}^d)$, $|\tilde{\Lambda}(\varphi)| = |\Lambda(\check{\varphi})| \underset{\exists C, N \text{ by P VII.18}}{\leq} C \cdot p_N(\check{\varphi}) = C\, p_N(\varphi)$

$\Rightarrow \tilde{\Lambda} \in \mathcal{S}'(\mathbb{R}^d)$, clearly $\tilde{\Lambda}|_{\mathcal{D}(\mathbb{R}^d)} = \check{\Lambda}$

Prop. VII.23 $\Lambda_n \to \Lambda$ in $\mathcal{S}'(\mathbb{R}^d)$

(a) $\forall \alpha \in \mathbb{N}_0^d$: $D^\alpha \Lambda_n \to D^\alpha \Lambda$

$$D^\alpha \Lambda_n(\varphi) = \underbrace{(-1)^{|\alpha|}\Lambda_n(D^\alpha\varphi) \to (-1)^{|\alpha|}\Lambda(D^\alpha\varphi)}_{\text{Prop. VII.22 (a)}} = D^\alpha\Lambda(\varphi)$$

(b) $f \in \mathcal{S}$ or f is a polynomial $\Rightarrow f\Lambda_n \to f\Lambda$

$$f\Lambda_n(\varphi) = \underbrace{\Lambda_n(f\varphi) \to \Lambda(f\varphi)}_{\text{Prop. VII.22 (b)}} = f\Lambda(\varphi)$$