$U \in \mathcal{D}'(\mathbb{R}^d)$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$

$$U * \varphi(x) = U(\tau_x \check{\varphi}) = U(y \mapsto \varphi(x-y)) \quad , x \in \mathbb{R}^d$$

(a) $f \in L^1_{loc}(\mathbb{R}^d) \Rightarrow \Lambda_f * \varphi = f * \varphi$

$$\ulcorner \Lambda_f * \varphi(x) = \Lambda_f(y \mapsto \varphi(x-y)) = \int_{\mathbb{R}^d} f(y)\varphi(x-y)\,dy = f * \varphi(x) \lrcorner$$

(b) $U * \varphi \in C^\infty(\mathbb{R}^d)$, $D^\alpha(U * \varphi) = D^\alpha U * \varphi = U * D^\alpha \varphi$

$\ulcorner$ • $U * \varphi$ is cts: $x_n \to x$ in $\mathbb{R}^d \overset{\text{Lemma VII.13(a)}}{\Rightarrow} \tau_{x_n}\check{\varphi} \to \tau_x \check{\varphi}$ in $\mathcal{D}(\mathbb{R}^d)$

$$\Rightarrow U(\tau_{x_n}\check{\varphi}) \to U(\tau_x \check{\varphi}) \Rightarrow U * \varphi(x_n) \to U * \varphi(x)$$

• $$\frac{\partial}{\partial x_j}(U * \varphi)(x) = \lim_{t \to 0} \frac{U * \varphi(x + t e_j) - U * \varphi(x)}{t} = \lim_{t \to 0} U\left(\frac{\tau_{x+te_j}\check{\varphi} - \tau_x \check{\varphi}}{t}\right)$$

$$\underset{\psi := \tau_x \check{\varphi}}{=} \lim_{t \to 0} U\left(\frac{\tau_{te_j}\psi - \psi}{t}\right) = \lim_{t \to 0} U\Big(y \mapsto \underbrace{\frac{\psi(y - te_j) - \psi(y)}{t}}_{\text{by Lemma VII.13 (b)} \; \partial_{-e_j}\psi = -D^{e_j}\psi}\Big)$$

$$= U(-D^{e_j}\psi) = U(y \mapsto D^{e_j}\varphi(x-y)) = U * D^{e_j}\varphi(x)$$

$\psi(y) = \varphi(x-y)$

$D^{e_j}\psi(y) = -D^{e_j}\varphi(x-y)$

and, moreover:

$$= D^{e_j}U(\psi) = D^{e_j}U(\tau_x \check{\varphi}) = (D^{e_j}U) * \varphi(x)$$

Hence: For $|\alpha| = 1$ the derivative exists and the formulas hold.

In particular, $D^\alpha(U * \varphi)$ is cts, so $U * \varphi \in C^1(\mathbb{R}^d)$

• general α by induction: Assume it holds for $|\alpha| \le k$

Let $|\alpha| = k+1$. Then $\alpha = \beta + e_j$, where $|\beta| = k$

$$D^\alpha(U * \varphi) = D^{e_j} D^\beta(U * \varphi) = D^{e_j}(U * D^\beta \varphi) = U * D^{\beta + e_j}\varphi = U * D^\alpha \varphi$$

$$= D^{e_j}(D^\beta U * \varphi) = D^{\beta + e_j} U * \varphi = D^\alpha U * \varphi$$

induction hypothesis

the case $|\alpha| = 1$

In particular, $D^\alpha(U*\varphi) = U*D^\alpha\varphi$ is cts by the first step
So, $U*\varphi$ is C^∞.

$\lrcorner$

(c) $\operatorname{spt}(U*\varphi) \subset \operatorname{spt} U + \operatorname{spt}\varphi$

$\ulcorner$ $U*\varphi(x) \neq 0 \Rightarrow U(\tau_x\check{\varphi}) \neq 0 \overset{\text{Prop. VII.12(c)}}{\Rightarrow} \operatorname{spt}(\tau_x\check{\varphi}) \cap \operatorname{spt} U \neq \emptyset \Rightarrow (*)$

what is $\operatorname{spt}(\tau_x\check{\varphi})$:

$\tau_x\check{\varphi}(y) = \varphi(x-y)$

so $y \in \operatorname{spt}\tau_x\check{\varphi} \Leftrightarrow x-y \in \operatorname{spt}\varphi \Leftrightarrow y \in x - \operatorname{spt}\varphi$

so, $\operatorname{spt}(\tau_x\check{\varphi}) = x - \operatorname{spt}\varphi$

so, $(*) \Rightarrow (x - \operatorname{spt}\varphi) \cap \operatorname{spt} U \neq \emptyset \Rightarrow x \in \operatorname{spt}\varphi + \operatorname{spt} U$ $\lrcorner$

<u>Corollary:</u> U has compact support $\Rightarrow U*\varphi \in \mathcal{D}(\mathbb{R}^d)$

(d) (h_j) smoothing kernel $\Rightarrow \Lambda_{U*h_j} \longrightarrow U$ in $\mathcal{D}'(\mathbb{R}^d)$

$\ulcorner$ $\Lambda_{U*h_j}(\varphi) = \int U*h_j(x)\,\varphi(x)\,dx = \int U(y \mapsto h_j(x-y))\cdot\varphi(x)\,dx$

$= \int U(y \mapsto \underbrace{\varphi(x)\,h_j(x-y)}_{\in\mathcal{D}(\mathbb{R}^d\times\mathbb{R}^d)})\,dx =$

it is C^∞; $\varphi(x)h_j(x-y) \neq 0 \Rightarrow x \in \operatorname{spt}\varphi,\ y \in x - \operatorname{spt} h_j \subset \operatorname{spt}\varphi - \operatorname{spt} h_j$.
so, the support is compact

$= \Lambda_1(x \longmapsto U(y \longmapsto \varphi(x)\,h_j(x-y))) \overset{\text{Prop. VII.14(5)}}{=}$

$= U(y \longmapsto \Lambda_1(x \longmapsto \varphi(x)\,h_j(x-y))) = U(y \mapsto \int \varphi(x)\,h_j(x-y)\,dx)$

$= U(\varphi * \check{h}_j)$

$(\check{h}_j)$ is also a smoothing kernel (it may be also chosen to be even, then $\check{h}_j = h_j$)

$\varphi * \check{h}_j \longrightarrow \varphi$ in $\mathcal{D}(\mathbb{R}^d)$

$\ulcorner$ $\operatorname{supp} \varphi * \check{h}_j \subset \operatorname{supp}\varphi + \operatorname{supp}\check{h}_j \subset \operatorname{supp}\varphi + U(0,1)$, so it is contained in one compact set

$\varphi * \check{h}_j \rightrightarrows \varphi$ (Theorem IV.6 (iii) ... φ is uniformly cts)

$D^\alpha(\varphi * \check{h}_j) = D^\alpha\varphi * \check{h}_j \rightrightarrows D^\alpha\varphi$ (the same)

$\uparrow$ Prop IV.3 $\lrcorner$

So, $\Lambda_{U * h_j}(\varphi) = U(\varphi * \check{h}_j) \to U(\varphi)$ and the proof is complete. $\lrcorner$

(e) $\tau_x(U * \varphi) = \tau_x U * \varphi = U * \tau_x\varphi$

$\ulcorner$ $\tau_x(U*\varphi)(z) = U*\varphi(z-x) = U(\tau_{z-x}\check{\varphi}) = U(\tau_z(\tau_{-x}\check{\varphi})) =$

$\ulcorner$ $\tau_{-x}\check{\varphi}(y) = \check{\varphi}(y+x) = \varphi(-x-y) = \tau_x(\varphi(-y)) = (\tau_x\varphi)^\vee(y)$ $\lrcorner$

$= U(\tau_z(\tau_x\varphi)^\vee) = U * \tau_x\varphi\ (z)$

It is the same

$(\tau_x U)*\varphi(z) = (\tau_x U)(\tau_z\check{\varphi}) = U(\tau_{-x}\tau_z\check{\varphi}) = U(\tau_{z-x}\check{\varphi}) = \tau_x(U*\varphi)(z)$ $\lrcorner$

(f) $U*(\varphi*\psi) = (U*\varphi)*\psi$

$\ulcorner$ $U*(\varphi*\psi)(x) = U(y \mapsto (\varphi*\psi)(x-y)) = U\left(y \mapsto \int\varphi(x-y-z)\psi(z)\,dz\right)$

$= U(y \mapsto \Lambda_1(z \mapsto \varphi(x-y-z)\psi(z))) \overset{\text{Prop VII 14 (b)}}{=}$

$= \Lambda_1(z \mapsto U(y \mapsto \varphi(x-y-z)\psi(z))) = \Lambda_1(z \mapsto \psi(z)U(y \mapsto \varphi(x-y-z)))$

$= \Lambda_1(z \mapsto \psi(z)(U*\varphi)(x-z)) = \int\psi(z)(U*\varphi)(x-z)\,dz = (U*\varphi)*\psi\ (x)$ $\lrcorner$