Let $\varphi \in \mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ $(d_1, d_2 \in \mathbb{N})$

(a) $\Lambda \in \mathcal{D}'(\mathbb{R}^{d_1})$ Define $\psi(y) = \Lambda(x \mapsto \varphi(x,y))$, $y \in \mathbb{R}^{d_2}$

Then $\psi \in \mathcal{D}(\mathbb{R}^{d_2})$ and $\forall \alpha : D^\alpha \psi(y) = \Lambda(x \mapsto D^{(0,\alpha)} \varphi(x,y))$

Proof: Fix $c > 0$ s.t. $\operatorname{spt} \varphi \subset \overline{U_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}}(0, c)}$

We proceed in several steps

① ψ is well defined:

for $y \in \mathbb{R}^{d_2}$ define $\varphi_y(x) = \varphi(x,y)$, $x \in \mathbb{R}^{d_1}$

- clearly $\varphi_y \in C^\infty(\mathbb{R}^{d_1})$ (as $\varphi \in C^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$)
- $\operatorname{spt} \varphi_y \subset \overline{U_{\mathbb{R}^{d_1}}(0,c)}$, so it is compact

We deduce that $\varphi_y \in \mathcal{D}(\mathbb{R}^{d_1})$ and hence $\psi(y) = \Lambda(\varphi_y)$ is well defined.

② $\operatorname{spt} \psi$ is compact: If $\|y\| > c$, necessarily $\varphi_y \equiv 0$, so $\psi(y) = \Lambda(0) = 0$.

Hence $\operatorname{spt} \psi \subset \overline{U_{\mathbb{R}^{d_2}}(0,c)}$

③ $y_n \to y$ in $\mathbb{R}^{d_2} \Rightarrow \varphi_{y_n} \to \varphi_y$ in $\mathcal{D}(\mathbb{R}^{d_1})$

- WLOG $\|y_n\| \leq c$ for all $n \in \mathbb{N}$

If $\|y_n\| > c$ for finitely many n, just omit them

If $\|y_n\| > c$ for all $n \in \mathbb{N}$ except for finitely many, then $\exists n_0 \; \forall n \geq n_0 : \varphi_{y_n} = 0$. Moreover, $\|y\| \geq c$, so $\varphi_y = 0$. Thus $\varphi_{y_n} \to \varphi_y$.

Otherwise we split (y_n) to two subsequences (y_{n_k}) and (y_{m_k}) covering whole sequence (y_n) s.t. $\|y_{n_k}\| > c$ and $\|y_{m_k}\| \leq c$

Then $\|y\| = c$ and hence $\varphi_{y_{n_k}} = 0 \to 0 = \varphi_y$

and it is enough to prove the convergence for $\varphi_{y_{m_k}}$

- So, assume $\forall n : \|y_n\| \leq c$ and, consequently, $\|y\| \leq c$.

We are going to show that $\varphi_{y_n} \to \varphi_y$ in this case

☐ $\operatorname{spt} \varphi_{y_n} \subset \overline{U_{\mathbb{R}^{d_1}}(0,c)}$ for each n (see ① above)

☐ Fix α multiindex and show that $D^\alpha \varphi_{y_n} \rightrightarrows D^\alpha \varphi_y$

Observe that $D^{\alpha}\varphi_{y_n}(x) = D^{(\alpha,0)}\varphi(x,y_n)$

$D^{(\alpha,0)}\varphi$ is cts on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$, so uniformly cts on $\overline{U_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}}(0,C)}$

Fix $\varepsilon > 0$. Then $\exists\, \delta > 0$ s.t.

$(u_1,v_1),(u_2,v_2) \in \overline{U_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}}(0,C)}$, $\|(u_1,v_1)-(u_2,v_2)\| < \delta \Rightarrow \|D^{(\alpha,0)}\varphi(u_1,v_1) - D^{(\alpha,0)}\varphi(u_2,v_2)\| < \varepsilon$

Fix $n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0$: $\|y_n - y\| < \delta$

If $n \geq n_0$ and $x \in \overline{U_{\mathbb{R}^{d_1}}(0,C)}$, then

$|D^{\alpha}\varphi_{y_n}(x) - D^{\alpha}\varphi_y(x)| = |D^{(\alpha,0)}\varphi(x,y_n) - D^{(\alpha,0)}\varphi(x,y)| < \varepsilon$

So, $\|D^{\alpha}\varphi_{y_n} - D^{\alpha}\varphi_y\|_{\infty} \leq \varepsilon$. This completes the proof of the convergence.

④ ψ is cts :

$y_n \to y$ in $\mathbb{R}^{d_2}$ $\overset{③}{\Longrightarrow}$ $\varphi_{y_n} \to \varphi_y$ in $\mathcal{D}(\mathbb{R}^{d_1})$ $\Rightarrow \psi(y_n) = \Lambda(\varphi_{y_n}) \to \Lambda(\varphi_y) = \psi(y)$

⑤ $\frac{\partial\psi}{\partial y_j}(y) = \Lambda\left(x \to \frac{\partial\varphi}{\partial y_j}(x,y)\right)$ $\quad$ ($j = 1,\dots,d_1$, $y \in \mathbb{R}^{d_2}$)

┌ $\frac{\partial\psi}{\partial y_j}(y) = \lim\limits_{t\to 0} \frac{\psi(y+t\cdot e_j) - \psi(y)}{t} = \lim\limits_{t\to 0} \frac{\Lambda(x\mapsto\varphi(x,y+te_j)) - \Lambda(x\mapsto\varphi(x,y))}{t}$

$e_j = (0\cdots,0,\underset{\overset{\uparrow}{j}}{1},0\cdots 0) \in \mathbb{R}^{d_2}$

$$\overset{\Lambda \text{ is linear}}{=} \lim_{t\to 0} \Lambda\left(x \mapsto \underbrace{\frac{\varphi(x,y+te_j)-\varphi(x,y)}{t}}_{\varphi_t(x,y)}\right) = (*)$$

$\varphi_t(x,y)$ using notation from Lemma VII.13 for $e = (0,e_j)$

Lemma VII.13 (b) $\Rightarrow$ $\varphi_t \to \partial_{(0,e_j)}\varphi = \frac{\partial}{\partial y_j}\varphi$ in $\mathcal{D}(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$

given $y \in \mathbb{R}^{d_2}$, it follows $(\varphi_t)_y \to \left(\frac{\partial\varphi}{\partial y_j}\right)_y$ in $\mathcal{D}(\mathbb{R}^{d_1})$

[obvious from definitions]

Therefore

$(*) = \Lambda\left(\left(\frac{\partial\varphi}{\partial y_j}\right)_y\right) = \Lambda\left(x \mapsto \frac{\partial\varphi}{\partial y_j}(x,y)\right)$ ┘

⑥ $\varphi \in C^\infty(\mathbb{R}^{d_2})$ and $\forall \alpha$: $D^\alpha \varphi(y) = \Lambda(x \mapsto D^{(0,\alpha)} \varphi(x,y))$

- ⑤ $\Rightarrow$ the formula holds if $|\alpha| = 1$
- ④ applied to $\frac{\partial \varphi}{\partial y_j} \in \mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ together with the formula implies $\varphi \in C^1(\mathbb{R}^{d_2})$
- we conclude by induction : Assume it holds if $|\alpha| \leq \ell$ for some $\ell \in \mathbb{N}$

Assume $|\alpha| = \ell + 1$. Then $\alpha = \beta + e_j$ for some $j \in \{1,..,d_2\}$ and β with $\beta = \ell$

Then for $y^0 \in \mathbb{R}^{d_2}$:

$$D^\alpha \varphi(y^0) = \frac{\partial}{\partial y_j}(D^\beta \varphi)(y^0) \underset{\substack{\uparrow \\ \text{induction} \\ \text{hypothesis}}}{=} \frac{\partial}{\partial y_j}\left(y \mapsto \Lambda(x \mapsto D^{(0,\beta)}\varphi(x,y))\right)\Big|_{y=y^0}$$

$$\underset{\substack{\uparrow \\ \text{⑤ applied to } D^{(0,\beta)}\varphi \in \mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})}}{=} \left(y \mapsto \Lambda\left(x \mapsto \frac{\partial}{\partial y_j}(D^{(0,\beta)}\varphi)(x,y)\right)\right)\Big|_{y=y^0} = \Lambda(x \mapsto D^{(0,\alpha)}\varphi(x,y^0))$$

Moreover, by ④ applied to $D^{(0,\beta)}\varphi$ we get that $D^\alpha \varphi$ is cts.

Conclusion: The statement follows by combining ⑥ and ②

Lemma R (representation of distributions)

$\Omega \subset \mathbb{R}^d$ open, $\Lambda \in \mathcal{D}'(\Omega)$, $K \subset \Omega$ compact

$\Rightarrow \exists N \in \mathbb{N}_0$, $\exists \mu_\alpha$, $|\alpha| \leq N$ regular Borel measures on K s.t.

$\forall \varphi \in \mathcal{D}_K(\Omega)$:

$$\Lambda(\varphi) = \sum_{|\alpha| \leq N} \int_K D^\alpha \varphi \, d\mu_\alpha$$

Proof: Prop VII.6 $\Rightarrow \exists C > 0, N \in \mathbb{N}_0$ s.t. $|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N$, $\varphi \in \mathcal{D}_K(\Omega)$

Let $X = (C(K))^{\{\alpha, |\alpha| \leq N\}}$ be equipped with the max-norm

$$\|(f_\alpha)_{|\alpha| \leq N}\| = \max_{|\alpha| \leq N} \|f_\alpha\|_\infty$$

Then X is a Banach space

Define $T: \mathcal{D}_K(\Omega) \longrightarrow X$ by $T(\varphi) = (D^\alpha \varphi|_K)_{|\alpha| \leq N}$

Then T is linear. Then $Y := T(\mathcal{D}_K(\Omega)) \subset\subset X$

Moreover, there is a linear mapping $L: Y \to \mathbb{F}$ such that

$L(T\varphi) = \Lambda(\varphi)$ for $\varphi \in \mathcal{D}_K(\Omega)$

[L correctly defin.: $T\varphi_1 = T\varphi_2 \Rightarrow T(\varphi_1 - \varphi_2) = 0 \Rightarrow \|\varphi_1 - \varphi_2\|_N = 0 \Rightarrow \Lambda(\varphi_1) = \Lambda(\varphi_2)$

L linear: $L(aT_\varphi + bT_\psi) = L(T(a\varphi + b\psi)) = \Lambda(a\varphi + b\psi) = a\Lambda(\varphi) + b\Lambda(\psi) =$

$= aL(T\varphi) + bL(T\psi)$]

Moreover, $|L(T\varphi)| = |\Lambda(\varphi)| \leq C\|\varphi\|_N = C \cdot \|T\varphi\|$,

so L is cts, $\|L\| \leq C$

H-B $\Rightarrow \exists\ \tilde{L}: X \to \mathbb{F}$ linear extension of L, $\|\tilde{L}\| = \|L\| \leq C$

Riesz representation theorem $\Rightarrow \exists\ (\mu_\alpha)_{|\alpha| \leq N}$ regular Borel measures on K s.t.

$$\tilde{L}((f_\alpha)_{|\alpha| \leq N}) = \sum_{|\alpha| \leq N} \int_K f_\alpha \, d\mu_\alpha$$

Then for $\varphi \in \mathcal{D}_K(\Omega)$:

$$\Lambda(\varphi) = L(T\varphi) = \tilde{L}(T\varphi) = \sum_{|\alpha| \leq N} \int_K D^\alpha \varphi \, d\mu_\alpha.$$

(b) [FUBINI THEOREM FOR DISTRIBUTIONS] $\Lambda_1 \in \mathcal{D}(\mathbb{R}^{d_1})$, $\Lambda_2 \in \mathcal{D}(\mathbb{R}^{d_2})$

$\varphi \in \mathcal{D}(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \Rightarrow$

$$\Lambda_2(y \longmapsto \Lambda_1(x \longmapsto \varphi(x,y))) = \Lambda_1(x \longmapsto \Lambda_2(y \longmapsto \varphi(x,y)))$$

<u>Proof</u>:

- $y \longmapsto \Lambda_1(x \longmapsto \varphi(x,y))$ belongs to $\mathcal{D}(\mathbb{R}^{d_2})$ } by (a)
  $x \longmapsto \Lambda_2(y \longmapsto \varphi(x,y))$ belongs to $\mathcal{D}(\mathbb{R}^{d_1})$
  so the expressions are well defined.

- Let $c > 0$ be such that $\operatorname{supp} \varphi \subset \overline{U_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}}(0,c)}$

Lemma R: $\Lambda_1, \overline{U_{\mathbb{R}^{d_1}}(0,c)} \rightsquigarrow N_1 \in \mathbb{N}_0, \mu_\alpha$ for $|\alpha| \leq N_1$

$\Lambda_2, \overline{U_{\mathbb{R}^{d_2}}(0,c)} \rightsquigarrow N_2 \in \mathbb{N}_0, \nu_\alpha$ for $|\alpha| \leq N_2$

$$\Lambda_2(y \mapsto \Lambda_1(x \mapsto \varphi(x,y))) = \sum_{|\beta| \leq N_2} \int D^{\beta} \Lambda_1(x \mapsto \varphi(x,y))\, d\nu_\beta(y)$$

$$\overset{(a)}{=} \sum_{|\beta| \leq N_2} \int \Lambda_1(x \mapsto D^{(\alpha,\beta)} \varphi(x,y))\, d\nu_\beta(y) = \sum_{|\beta| \leq N_2} \int \sum_{|\alpha| \leq N_1} \int D^{(\alpha,\beta)} \varphi(x,y)\, d\mu_\alpha(x)\, d\nu_\beta(y)$$

$$= \sum_{|\beta| \leq N_2} \sum_{|\alpha| \leq N_1} \int\int D^{(\alpha,\beta)} \varphi(x,y)\, d\mu_\alpha(x)\, d\nu_\beta(y)$$

$$\overset{\text{FUBINI}}{=}$$

$D^{(\alpha,\beta)}\varphi$ is bdd, cts

$$= \sum_{|\beta| \leq N_2} \sum_{|\alpha| \leq N_1} \int\int D^{(\alpha,\beta)} \varphi(x,y)\, d\nu_\beta(y))\, d\mu_\alpha(x)$$

$$= \sum_{|\alpha| \leq N_1} \int \sum_{|\beta| \leq N_2} \int D^{(\alpha,\beta)} \varphi(x,y)\, d\nu_\beta(y))\, d\mu_\alpha(x)$$

$$= \sum_{|\alpha| \leq N_1} \int \Lambda_2(y \mapsto D^{(\alpha,0)} \varphi(x,y))\, d\mu_\alpha(x)$$

$$\overset{(a)}{=} \sum_{|\alpha| \leq N_1} \int D^{\alpha} \Lambda_2(y \mapsto \varphi(x,y))\, d\mu_\alpha(x)$$

$$= \Lambda_1(x \mapsto \Lambda_2(y \mapsto \varphi(x,y)))$$