(a) $x_n \to x$ in $\mathbb{R}^d$, $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow \tau_{x_n}\varphi \to \tau_x \varphi$

Γ Recall $\tau_x \varphi(y) = \varphi(y-x)$, $y \in \mathbb{R}^d$

- $\varphi \in \mathcal{D}(\Omega) \Rightarrow \exists\ r_1 > 0$ s.t. $\operatorname{spt}\varphi \subset U(0,r_1)$

  $x_n \to x \Rightarrow \exists\ r_2 > 0$ s.t. $\{x_n, n \in \mathbb{N}\} \subset U(0,r_2)$

  $K := \overline{U(0, r_1+r_2)} \Rightarrow K \subset \mathbb{R}^d$ is compact

  and $\forall n \in \mathbb{N}$ $\operatorname{spt}\tau_{x_n}\varphi \subset K$ and also $\operatorname{spt}\tau_x\varphi \subset K$

  Γ $\tau_{x_n}\varphi(y) \neq 0 \Rightarrow \varphi(y-x_n) \neq 0 \Rightarrow y - x_n \in \operatorname{spt}\varphi$

  $\Rightarrow y \in \operatorname{spt}\varphi + x_n \subset U(0,r_1) + U(0,r_2) \subset K$

  Similarly,

  $\tau_x\varphi(y) \neq 0 \Rightarrow \varphi(y-x) \neq 0 \Rightarrow y - x \in \overline{\operatorname{spt}\varphi}$

  $\Rightarrow y \in \operatorname{spt}\varphi + x \subset U(0,r_1) + \overline{U(0,r_2)} \subset K$ $\rfloor$

- α multiindex $\Rightarrow \|D^\alpha \tau_{x_n}\varphi - D^\alpha \tau_x \varphi\|_\infty = \sup\limits_{y \in \mathbb{R}^d} |D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)|$

  $= \sup\limits_{y \in K} |D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)|$

  $y \in K \Rightarrow y - x_n, y - x \in K + \overline{U(0,r_2)} = \overline{U(0, r_1+2r_2)}$

  this is compact. $D^\alpha\varphi$ is cts, so uniformly cts on $\overline{U(0,r_1+2r_2)}$

  So, given $\varepsilon > 0$, there is $\delta > 0$ s.t.

  $\forall y_1, y_2 \in \overline{U(0, r_1+2r_2)} : \|y_1 - y_2\| < \delta \Rightarrow |D^\alpha\varphi(y_1) - D^\alpha\varphi(y_2)| < \varepsilon$

  Further, $x_n \to x$, so there is $n_0 \in \mathbb{N}$ s.t. $\forall n \geq n_0 : \|x_n - x\| < \delta$

  Now, given $n \geq n_0$ and $y \in K$, we have $y - x_n, y - x \in \overline{U(0,r_1+2r_2)}$,

  $\|(y-x_n) - (y-x)\| = \|x - x_n\| < \delta$ and hence

  $|D^\alpha\varphi(y-x_n) - D^\alpha\varphi(y-x)| < \varepsilon$

  Take sup over $y \in K$ and deduce that

  $\|D^\alpha \tau_{x_n}\varphi - D^\alpha\tau_x\varphi\|_\infty \leq \varepsilon$ for $n \geq n_0$

  This completes the proof that $D^\alpha \tau_{x_n}\varphi \rightrightarrows D^\alpha\tau_x\varphi$.

(b1): $\varphi \in \mathcal{D}(\mathbb{R}^d),\ e \in \mathbb{R}^d \Rightarrow \partial_e \varphi \in \mathcal{D}(\mathbb{R}^d)$

$\ulcorner$ Recall $\partial_e \varphi(x) = \lim_{t \to 0} \frac{\varphi(x+te) - \varphi(x)}{t}$, $x \in \mathbb{R}^d$

Fix $x \in \mathbb{R}^d$. Define $g_x(t) = \varphi(x+te)$, $t \in \mathbb{R}$. Then $g_x \in C^\infty(\mathbb{R})$

and $\partial_e \varphi(x) = g_x'(0) = \sum_{j=1}^d \frac{\partial \varphi}{\partial x_j}(x+te) \cdot e_j \Big|_{t=0} = \sum_{j=1}^d \frac{\partial \varphi}{\partial x_j}(x) e_j$

So, $\partial_e \varphi = \sum_{j=1}^d e_j \cdot \frac{\partial \varphi}{\partial x_j} \in \mathcal{D}(\mathbb{R}^d)$ $\lrcorner$

(b2) For $r \in \mathbb{R} \setminus \{0\}$ define $\varphi_r(x) = \frac{1}{r}\left(\varphi(x+re) - \varphi(x)\right)$.

Clearly $\varphi_r \in \mathcal{D}(\mathbb{R}^d)$. We claim that $\varphi_r \xrightarrow{r \to 0} \partial_e \varphi$ in $\mathcal{D}(\mathbb{R}^d)$

$\ulcorner$ Fix $c > 0$ s.t. $\operatorname{spt} \varphi \subset U(0,c)$. If $0 < |r| < 1$, then

$\operatorname{spt} \tau_{re} \varphi \subset \operatorname{spt} \varphi + U(0, \|e\|) \subset \overline{U(0, c + \|e\|)}$

so, supports are contained in a unique compact.

Further, given $r \in \mathbb{R}$, $0 < |r| < 1$ and $x \in \mathbb{R}^d$ we have

$$|\varphi_r(x) - \partial_e \varphi(x)| = \left| \frac{1}{r}\left(g_x(0) - g_x(-r)\right) - g_x'(0) \right| =$$

$$= \left| \frac{1}{r} \int_0^r g_x'(t)\,dt - g_x'(0) \right| = \left| \frac{1}{r} \int_0^r \left(g_x'(t) - g_x'(0)\right) dt \right| =$$

$$= \left| \frac{1}{r} \int_0^r \underbrace{\sum_{j=1}^d e_j \cdot \left(\frac{\partial \varphi}{\partial x_j}(x+te) - \frac{\partial \varphi}{\partial x_j}(x) \right)} dt \right| \leq$$

$$|\ \cdot\ | \leq \|e\| \cdot \left(\sum_{j=1}^d \left| \frac{\partial \varphi}{\partial x_j}(x+te) - \frac{\partial \varphi}{\partial x_j}(x) \right|^2 \right)^{1/2}$$

Cauchy-Schwarz

$$\leq \|e\| \cdot \left(\sum_{j=1}^d \left\| \tau_{-te} \frac{\partial \varphi}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \right\|^2 \right)^{1/2}$$

$\varepsilon > 0 \overset{(a)}{\Longrightarrow} \exists\, \delta > 0\ \forall j \in \{1, \dots, d\}\ \forall y : \|y\| < \delta : \Rightarrow \left\| \tau_y \frac{\partial \varphi}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \right\|_\infty < \varepsilon$

So, if $0 < |r| \|e\| < \delta$, then $\leq \frac{1}{|r|} \cdot |r| \cdot \|e\| \cdot \sqrt{d} \cdot \varepsilon$

Conclusion: If $0 < |h| < \delta$, then $\| \varphi_h - \partial_e \varphi \| < \|e\| \sqrt{d} \cdot \varepsilon$

We deduce that $\varphi_h \rightrightarrows \partial_e \varphi$

Now, assume α is a multiindex. Then

$$D^\alpha \varphi_h = (D^\alpha \varphi)_h \rightrightarrows \partial_e (D^\alpha \varphi) = D^\alpha (\partial_e \varphi)$$

apply the above arguments to $D^\alpha \varphi \in \mathcal{D}(\mathbb{R}^d)$.