$\Omega \subset \mathbb{R}^d$ open, $\Lambda \in \mathcal{D}'(\Omega)$, $f \in C^\infty(\Omega) \Rightarrow f\Lambda \in \mathcal{D}'(\Omega)$

① Recall that $f\Lambda(\varphi) = \Lambda(f\varphi)$, $\varphi \in \mathcal{D}(\Omega)$
since $f \in C^\infty(\Omega)$, $\varphi \in \mathcal{D}(\Omega) \Rightarrow f \cdot \varphi \in \mathcal{D}(\Omega)$, $f\Lambda$ is well defined,
clearly it is a linear mapping $\mathcal{D}(\Omega) \to \mathbb{C}$

② To prove $f\Lambda \in \mathcal{D}'(\Omega)$ fix $K \subset \Omega$ compact
Since $\Lambda \in \mathcal{D}'(\Omega)$, there are $C > 0$ and $N \in \mathbb{N}_0$ s.t.

$$|\Lambda(\varphi)| \leq C \cdot \|\varphi\|_N, \quad \varphi \in \mathcal{D}_K(\Omega)$$

Fix now $\varphi \in \mathcal{D}_K(\Omega)$ and estimate

$$|f\Lambda(\varphi)| = |\Lambda(f\varphi)| \leq C \cdot \|f\varphi\|_N \qquad (\text{note that } f\varphi \in \mathcal{D}_K(\Omega))$$

Fix $\alpha \in \mathbb{N}_0^d$, $|\alpha| \leq N$. Then for $x \in K$:

$$|D^\alpha(f\varphi)(x)| = \Big|\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta f(x)\, D^{\beta-\alpha}\varphi(x)\Big| \leq$$

$$\left(\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdot \binom{\alpha_2}{\beta_2} \cdots \binom{\alpha_d}{\beta_d} \right)$$

$$\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|D^\beta f|_K\|_\infty \, \|D^{\beta-\alpha}\varphi\|_\infty \leq \|f|_K\|_N \cdot \|\varphi\|_N \cdot \sum_{\beta \leq \alpha} \binom{\alpha}{\beta}$$

$$\sum_{\beta \leq \alpha} \binom{\alpha}{\beta} = \sum_{\beta_1=0}^{\alpha_1} \sum_{\beta_2=0}^{\alpha_2} \cdots \sum_{\beta_d=0}^{\alpha_d} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_d}{\beta_d} = \left(\sum_{\beta_1=0}^{\alpha_1} \binom{\alpha_1}{\beta_1}\right)\left(\sum_{\beta_2=0}^{\alpha_2} \binom{\alpha_2}{\beta_2}\right) \cdots \left(\sum_{\beta_d=0}^{\alpha_d} \binom{\alpha_d}{\beta_d}\right)$$

$$= 2^{\alpha_1} \cdot 2^{\alpha_2} \cdots 2^{\alpha_d} = 2^{|\alpha|} \leq 2^N$$

So, $\|f\varphi\|_N \leq 2^N \cdot \|f|_K\|_N \cdot \|\varphi\|_N$

Thus $|f\Lambda(\varphi)| \leq C \cdot 2^N \cdot \|f|_K\|_N \cdot \|\varphi\|_N$ for $\varphi \in \mathcal{D}_K(\Omega)$

hence $f\Lambda \in \mathcal{D}'(\Omega)$