Theorem VII.4 Let $f \in L^1_{loc}((a,b))$

(a) The weak derivative is uniquely determined:

$g_1, g_2 \in L^1_{loc}((a,b))$ two weak derivatives of f.

Then $\forall \varphi \in \mathcal{D}((a,b))$:

$$\int_a^b g_1\varphi = -\int_a^b f\varphi' = \int_a^b g_2\varphi \quad \text{, hence} \quad \int_a^b (g_1-g_2)\varphi = 0$$

By L2 we deduce $g_1 - g_2 = 0$ a.e.

μ_1, μ_2 two measures which are weak derivatives of f

Then $\forall \varphi \in \mathcal{D}((a,b))$

$$\int_a^b \varphi \, d\mu_1 = -\int_a^b f\varphi' = \int_a^b \varphi \, d\mu_2 \text{, hence } \int_a^b \varphi \, d(\mu_1-\mu_2) = 0$$

By L2 we get $\mu_1 - \mu_2 = 0$

(b1) f absolutely continuous on $[a,b]$

$\Rightarrow$
- f' exists a.e. and $f' \in L^1(a,b)$ [properties of AC functions]
- Integration by parts for AC functions yields

$$\int_a^b f'\varphi = \underbrace{[f\varphi]_a^b}_{=0} - \int_a^b f \cdot \varphi' \quad \Rightarrow f' \text{ is the weak derivative of } f$$

(b2) $g \in L^1((a,b))$ is the weak derivative of f

$\Rightarrow$ define $f_0(t) = \int_a^t g$, $t \in [a,b]$.

Then $f_0 \in AC[a,b]$, $f_0' = g$ a.e. [by properties of indefinite Lebesgue integral]

Hence by (b1) we know that g is the weak derivative of f_0

Thus $f - f_0$ has weak derivative — the constant zero function

By P3 we know $f - f_0$ is constant (i.e. $\exists C \in \mathbb{F}$ $f - f_0 = C$ a.e. thus $f = f_0 + C$ a.e.)

Therefore f equals a.e. to a AC function.

(b3) f locally absolutely continuous on (a,b), i.e. f is AC on $[c,d]$ for each $[c,d] \subset (a,b)$.

Then f' exist a.e., $f' \in L^1_{loc}$ and f' is the weak derivative of f

⌈apply (b1) to each $[c,d] \subset (a,b)$ and observe that $\forall \varphi \in \mathcal{D}((a,b))\ \exists [c,d] \subset (a,b): \varphi \in \mathcal{D}((c,d))$⌋

(b4) $g \in L^1_{loc}((a,b))$ is the weak derivative of f:

Fix $x_0 \in (a,b)$ and define $f_0(t) = \int_{t_0}^{t} g$, $t \in (a,b)$

Then $f_0 \in AC_{loc}((a,b))$

and $\exists c \in \mathbb{F}: f = f_0 + c$ a.e.

⌈similarly as in (b2)⌋

(c1) $\mu \geq 0$, non-negative measure which is a weak derivative of f:

Set $f_1(t) = \mu((a,t))$, $t \in (a,b)$

$\Rightarrow f_1$ is a non-decreasing function on (a,b) (and bdd)

$$\varphi \in \mathcal{D}((a,b)) \Rightarrow \int_{(a,b)} \varphi \, d\mu = \int_{(a,b)} \int_a^t \varphi'(s)\, ds\, d\mu(t) \underset{\substack{\varphi' \text{ cts} \\ \text{bdd}}}{\overset{\text{FUBINI}}{=}} \quad \left[a < s \leq t < b \right]$$

$$= \int_a^b \int_{[s,b)} \varphi'(s)\, d\mu(t)\, ds = \int_a^b \varphi'(s) \cdot \mu([s,b))\, ds = \int_a^b \varphi'(s) \left(\mu((a,b)) - \mu((a,s)) \right) ds$$

$$= \mu((a,b)) \cdot \underbrace{\int_a^b \varphi'}_{= [\varphi]_a^b = 0} - \int_a^b \varphi'(s) \underbrace{\mu((a,s))}_{= f_1(s)}\, ds = - \int_a^b f_1 \cdot \varphi'$$

$\Rightarrow \mu$ is the weak derivative of f_1

$\Rightarrow f - f_1$ has 0 as weak derivative

$\underset{P3}{\Rightarrow} \exists c \in \mathbb{R}: \ f - f_1 = c$ a.e., i.e. $f = f_1 + c$ a.e.

Thus f is a.e.-equal to a non-decreasing function.

c(2) f non-decreasing, bdd

For $(c,d) \subset (a,b)$ set $\mu((c,d)) = \lim_{t \to d-} f(t) - \lim_{t \to c+} f(t)$

and for $A \subset (a,b)$ arbitrary set

$$\mu^*(A) = \inf \{ \sum_n \mu(I_n) ;\ I_n \text{ open intervals } \subset (a,b),\ \bigcup_n I_n \supset A \}$$

$\Rightarrow \mu^*$ is a finite outer measure:

- $\mu^*(\emptyset) = 0$ [$\exists\, t_0$ a point of continuity of f. Then $\mu((t_0 - \delta, t_0 + \delta)) \to 0$ for $\delta \to 0+$]
- μ^* finite ..... $\mu^*(A) \le \mu((a,b))$
- μ^* σ-subadditive [$A = \bigcup_n A_n$, $\varepsilon > 0$

  $\exists\ (I_{n,k})$ open intervals $A_n \subset \bigcup_k I_{n,k}$

  $\sum_k \mu(I_{n,k}) < \mu^*(A_n) + \frac{\varepsilon}{2^n}$

  $\Rightarrow A \subset \bigcup_{n,k} I_{n,k}$ & $\sum_{n,k} \mu(I_{n,k}) < \sum_n (\mu^*(A_n) + \frac{\varepsilon}{2^n})$

  $= \sum_n \mu^*(A_n) + \varepsilon$

  $\Rightarrow \mu^*(A) \le \sum_n \mu^*(A_n) + \varepsilon$, $\varepsilon > 0$ arbitrary]

Moreover, $[c,d] \subset (a,b) \Rightarrow \mu^*([c,d]) = \lim_{t \to d+} f(t) - \lim_{t \to c-} f(c)$

[$\le$ $\varepsilon > 0 \Rightarrow \mu^*([c,d]) \le \mu((c-\varepsilon, d+\varepsilon)) \le f(d+\varepsilon) - f(c-\varepsilon)$, take $\varepsilon \to 0+$

$\ge$: I_k, $k \in \mathbb{N}$ open intervals, $[c,d] \subset \bigcup_k I_k$

$[c,d]$ compact $\Rightarrow \exists\, n \in \mathbb{N}$: $[c,d] \subset \bigcup_{k=1}^n I_k$

WLOG $I_k \cap [c,d] \ne \emptyset$ for $k \in \mathbb{N}$

Then $\sum_{k=1}^\infty \mu(I_k) \ge \sum_{k=1}^n \mu(I_k)$

Further, $\bigcup_{k=1}^n I_k = I$ is an open interval

Claim: $\mu(I) \le \sum_{k=1}^n \mu(I_k)$

Γ by induction: Assume $I_1 \cup \dots \cup I_n$ is an open interval c
Then $\mu(I_1 \cup \dots \cup I_n) \le \mu(I_1) + \dots + \mu(I_n)$

- $n=1$ ... obvious
- $n=2$: $I_1 \cup I_2$ open interval $\Rightarrow \mu(I_1 \cup I_2) \le \mu(I_1) + \mu(I_2)$

  Γ $I_1 \supset I_2$ ... clear

  $I_2 \supset I_1$ ... clear

  $I_1 = (\alpha, \beta),\ I_2 = (\gamma, \delta)$, $\alpha < \gamma < \beta < \delta$

  $\mu(I_1 \cup I_2) = \lim_{t \to \delta-} f(t) - \lim_{t \to \alpha+} f(t)$

  $\mu(I_1) + \mu(I_2) = \underbrace{\lim_{t \to \beta-} f(t) - \lim_{t\to\gamma+} f(t)}_{\ge 0 \text{ as } f \text{ is non-decreasing}} - \lim_{t \to \alpha+} f(t) + \lim_{t \to \delta-} f(t)$ $\checkmark$

- Assume it holds for some $n \ge 2$
  and let us have $I_1, \dots, I_{n+1}$

  Let $j \in \{1, \dots, n\}$ be such that $I_j \cap I_{n+1} \ne 0$

  Then $\mu(I_1 \cup \dots \cup I_{n+1}) \underset{\text{induction hypothesis}}{\le} \sum_{\substack{1 \le i \le n \\ i \ne j}} \mu(I_i) + \underbrace{\mu(I_j \cup I_{n+1})}_{\le \mu(I_j) + \mu(I_{n+1}) \text{ by the case } n=2} \le \sum_{i=1}^{n+1} \mu(I_i)$ $\checkmark$

Thus $\sum_{k=1}^{\infty} \mu(I_k) \ge \sum_{k=1}^{n} \mu(I_k) \ge \mu(I_1 \cup \dots \cup I_n) =$

$= \mu((\alpha, \beta)) = \lim_{t \to \beta-} f(t) - \lim_{t \to \alpha+} f(t) \ge \lim_{t \to d+} f(t) - \lim_{t \to c-} f(t)$

$\uparrow (\alpha, \beta) := I_1 \cup \dots \cup I_n$ $\lrcorner$

Next: $[c,d] \subset (a,b) \Rightarrow \mu^*((c,d)) = \mu((c,d))$

Γ $\le$: clear from definitions

$\ge$: $\varepsilon > 0$ small $\Rightarrow \mu^*((c,d)) \ge \mu^*([c+\varepsilon, d-\varepsilon]) = \lim_{t \to d-\varepsilon+} f(t) - \lim_{t \to c+\varepsilon-} f(t)$

$\ge f(d-\varepsilon) - f(c+\varepsilon) \xrightarrow{\varepsilon \to 0+} \mu((c,d))$ $\lrcorner$

Next step: $\forall A \subset (a,b)\ \forall c \in (a,b) : \mu^*(A) = \mu^*(A \cap (a,c]) + \mu^*(A \cap (c,b))$

Γ $\leq$ by σ-subadditivity

$\geq$ Let $\varepsilon > 0$, $I_n, n \in \mathbb{N}$ open intervals, $A \subset \bigcup_n I_n$, $\sum_n \mu(I_n) < \mu^*(A) + \varepsilon$

$J_n := I_n \cap (a,c]$, $H_n := I_n \cap (c,b)$

$A \cap (c,b) \subset \bigcup_n H_n \Rightarrow \mu^*(A \cap (c,b)) \leq \sum_n \mu(H_n)$

$A \cap (a,c] \subset \bigcup_n J_n \Rightarrow \mu^*(A \cap (a,c]) \leq \sum_n \mu^*(J_n)$

But $\mu(H_n) + \mu^*(J_n) = \mu(I_n)$:

$\left(\mu^*((u,v]) = \lim_{t \to v+} f(t) - \lim_{t \to u+} f(t) \right.$

$\geq$: $(u,v] \supset [u+\varepsilon, v]$ and use the previous steps

$\leq$ $(u,v] \subset (u, v+\varepsilon)$ $\left.\right)$

Thus $\mu^*(A \cap (c,b)) + \mu^*(A \cap (a,c]) \leq \sum \mu(H_n) + \sum \mu^*(J_n)$

$= \sum \mu(I_n) < \mu^*(A) + \varepsilon$

$\varepsilon > 0$ arbitrary $\Rightarrow$ we are done $\lrcorner$

Conclusion: By the Caratheodory construction μ^* restricted to the Borel σ-algebra is σ-additive. By definition it is regular

Set $f_1(t) = \mu((a,t))$, $t \in (a,b)$.

By (c1) μ is the weak derivative of f_1.

Since $f_1 = f$ except for a countable set, μ is the weak derivative of f

(c3) f of bdd variation

$\Rightarrow$ $\operatorname{Re} f$, $\operatorname{Im} f$ of bdd variation

f real valued $\Rightarrow f = f_1 - f_2$, f_1, f_2 bdd non-decreasing

so, by (c2) we deduce that there is a measure μ which is the weak derivative of f and $\mu((c,d)) = \lim_{t \to d-} f(t) - \lim_{t \to c+} f(t)$ for $(c,d) \subset (a,b)$

c(4) μ a signed or complex measure which is the weak derivative of f

Then $f_1(t) = \mu((a,t))$, $t \in (a,b)$, is of bdd variation

[μ real-valued --- take $\mu = \mu^+ - \mu^-$

μ complex --- $\mu = (\operatorname{Re}\mu)^+ - (\operatorname{Re}\mu)^- + i((\operatorname{Im}\mu)^+ - (\operatorname{Im}\mu)^-)$]

and μ is its weak derivative $\Rightarrow \exists c \in \mathbb{F}$ $f = f_1 + c$ a.e. //