Let X be a vector space (over $\mathbb{F}$) and let $\mathcal{U}$ be a nonempty family of subsets of X with properties:

(i) Elements of $\mathcal{U}$ are absolutely convex and absorbing

(ii) $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U} : (V+V=)\, 2V \subset U$

(iii) $\forall U, V \in \mathcal{U}\ \exists W \in \mathcal{U} : W \subset U \cap V$

Then there is a unique topology $\mathcal{T}$ on X s.t. $(X, \mathcal{T})$ is a LCS and $\mathcal{U}$ is a base of nbhds of 0 in $(X,\mathcal{T})$.

Moreover, $(X,\mathcal{T})$ is Hausdorff iff $\bigcap \mathcal{U} = \{0\}$

Proof: step 1: Uniqueness and formula for $\mathcal{T}$.

Since any linear topology is translation invariant, there is at most one linear topology with a given base of nbhds of 0. Therefore, uniqueness is clear. Moreover, by the definition of a base of nbhds, the topology $\mathcal{T}$ must be given by

$$\mathcal{T} = \{ G \subset X ;\ \forall x \in G\ \exists U \in \mathcal{U} : x + U \subset G \}$$

So, let us define $\mathcal{T}$ by this formula

Step 2: $\mathcal{T}$ is a topology on X:

- Clearly $\emptyset \in \mathcal{T}$, $X \in \mathcal{T}$
- $(G_\alpha)_{\alpha\in\Lambda}$ any family of elements of $\mathcal{T}$

$$\Rightarrow \bigcup_{\alpha\in\Lambda} G_\alpha \in \mathcal{T} \quad \Big\lceil x \in \bigcup_{\alpha\in\Lambda} G_\alpha \Rightarrow \exists\, \alpha : x \in G_\alpha, \text{ so}$$

$$\exists U \in \mathcal{U}\quad x + U \subset G_\alpha \subset \bigcup_{\alpha\in\Lambda} G_\alpha \Big\rfloor$$

- $G_1, G_2 \in \mathcal{T} \Rightarrow G_1 \cap G_2 \in \mathcal{T}$

$\Big\lceil x \in G_1 \cap G_2 \Rightarrow \begin{cases} x \in G_1 \Rightarrow \exists\, U \in \mathcal{U} : x + U \subset G_1 \\ x \in G_2 \Rightarrow \exists\, V \in \mathcal{U} : x + V \subset G_2 \end{cases}$ by (iii) find $W \in \mathcal{U}$ with $W \subset U \cap V$

Then $x + W \subset G_1 \cap G_2 \Big\rfloor$

Step 3: Any $V \in \mathcal{U}$ is a nbhd of 0 in $\mathcal{T}$; hence $\mathcal{U}$ is a base of nbhds of 0 in $\mathcal{T}$

$\ulcorner U \in \mathcal{U}$ ... Let $V := \{x \in U \,;\, \exists W \in \mathcal{U} : x + W \subset U\}$

Then clearly $\cdot\ V \subset U$

$\cdot\ 0 \in V$ (take $W = U$)

Moreover, $V \in \mathcal{T}$, i.e., V is open:

$\ulcorner$

$x \in V \Rightarrow$ by the definition of V $\exists W \in \mathcal{U}$ with $x + W \subset U$

by c(i) find $\tilde{W} \in \mathcal{U}$ with $2\tilde{W} \subset W$

Then $x + \tilde{W} \subset V$, as $\forall w \in \tilde{W}$

$x + w + \tilde{W} \subset x + \tilde{W} + \tilde{W} = x + 2\tilde{W} \subset x + W \subset U$ $\lrcorner$

So, $0 \in \operatorname{int} V$, hence V is a nbhd of 0 in $\mathcal{T}$. $\lrcorner$

Step 4: Addition is continuous in $\mathcal{T}$:

$\ulcorner x, y \in X$, $G \in \mathcal{T}$, $x + y \in G$

$\Rightarrow \exists U \in \mathcal{U}$ $x + y + U \subset G$ $\overset{c(i)}{\Rightarrow} \exists V \in \mathcal{U} : V + V = 2V \subset U$

Then $(x + V) + (y + V) \subset x + y + U \subset G$. $\lrcorner$

Step 5: U is a nbhd of 0 in $\mathcal{T}$, $\lambda > 0 \Rightarrow \lambda U$ is also a nbhd of 0 in $\mathcal{T}$

$\ulcorner$ $\lambda \geq 1$: U nbhd of 0 $\Rightarrow \exists V \in \mathcal{U} : V \subset U$

$V \subset \lambda V \subset \lambda U$ $\Big\}$ $\Rightarrow \lambda U$ is a nbhd of 0

$\uparrow$ V is balanced by (i)

$\lambda = \frac{1}{2}$: U nbhd of 0 $\overset{(ii)}{\Rightarrow} \exists V \in \mathcal{U}$: $2V \subset U$

Then $V \subset \frac{1}{2} U$

so $\frac{1}{2} U$ is a nbhd of 0

By induction: U is a nbhd of 0 $\Rightarrow \forall n \in \mathbb{N}$ $\frac{1}{2^n} U$ is a nbhd of 0

Hence general $\lambda > 0$: Find $n \in \mathbb{N}$ with $\frac{1}{2^n} < \lambda$. Then

$\lambda U = (\lambda \cdot 2^n) \cdot \frac{1}{2^n} U$ is a nbhd of 0 whenever U is a nbhd of 0 $\lrcorner$

<u>Step 6</u>: Continuity of multiplication

$\ulcorner$ $\lambda \in \mathbb{F}$, $x \in X$, $G \in \mathcal{J}$ s.t. $\lambda x \in G$

Find $U \in \mathcal{U}$ s.t. $\lambda x + U \subset G$

By (iii) $\exists V \in \mathcal{U}$ with $V + V \subset U$

By (i) V is absorbing, so $\exists c > 0$ $\forall t \in [0,c]$: $tx \in V$

Fix such c

V balanced $\Rightarrow \forall$ $t \in \mathbb{F}$, $|t| \leq c$: $tx \in V$

Let $\mu \in \mathbb{F}$, $y \in X$ be such that $|\mu - \lambda| < c$ & $y \in x + \frac{1}{|\lambda|+c} V$

(this defines a nbhd of (λ, x) in $\mathbb{F} \times (X, \mathcal{J})$ by definition and by Step 5). Then

$$\mu y - \lambda x = \underbrace{\mu(y-x)}_{\in \frac{\mu}{|\lambda|+c} V \subset V} + \underbrace{(\mu - \lambda) x}_{\in V, \text{ as } |\mu-\lambda| < c} \in V + V \subset U$$

as $\left|\frac{\mu}{|\lambda|+c}\right| \leq \frac{|\lambda| + |\mu - \lambda|}{|\lambda| + c} < 1$ and V is balanced

So, $\mu y \in \lambda x + U \subset G$. $\lrcorner$

Moreover part: $(X,\mathcal{T})$ a LCS, $\mathcal{U}$ a base of nbhds of 0

Then X is Hausdorff $\iff \bigcap \mathcal{U} = \{0\}$

$\Gamma \Rightarrow$: $x \in X \setminus \{0\} \Rightarrow \exists\ G_1, G_2 \in \mathcal{T}\ x \in G_1, 0 \in G_2,\ G_1 \cap G_2 = \emptyset$

Find $U \in \mathcal{U}$ with $U \subset G_2$. Then $x \notin U$

Hence $x \notin \bigcap \mathcal{U}$

$\Leftarrow$ $x, y \in X,\ x \neq y \Rightarrow x - y \neq 0$.

Find $U \in \mathcal{U}$ with $x - y \notin U$

Find V, an $\underbrace{\text{nbhd}}$ of 0 s.t. $V + V \subset U$ (absolutely convex)

Then $(x+V) \cap (y+V) = \emptyset$

Γ $z \in (x+V) \cap (y+V)$

$\Rightarrow z = x + v_1 = y + v_2,\ v_1, v_2 \in V$

$V = -V$

$\Rightarrow x - y = v_1 - v_2 \in V + V \subset U$ $\lightning$ $\lrcorner$