FUNCTIONAL ANALYSIS 1

WINTER SEMESTER 2016/2017

PROBLEMS TO CHAPTER VIII

Problems to Section VIII.1 – examples of Banach algebras, invertible elements

Problem 1. Let $A = (\mathbb{C}^n, \|\cdot\|_\infty)$, where $n \geq 2$.

(1) Define multiplication on A by
$$(x_1, x_2 \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = (x_1y_1, x_1y_2, \dots, x_1y_n).$$
Show that A equipped with this mutliplication is a Banach algebra and that A has many left units but no right unit.

(2) Define multiplication on A by
$$(x_1, x_2 \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = (x_1y_1, x_2y_1, \dots, x_2y_1).$$
Show that A equipped with this mutliplication is a Banach algebra and that A has many right units but no leftt unit.

Problem 2. Let $A = (\mathbb{C}^n, \|\cdot\|_p)$, where $p \in [1, \infty]$ and $n \geq 2$. Equip A with the coordinatewise multiplication, i.e.,
$$(x_1, x_2 \dots, x_n) \cdot (y_1, y_2 \dots, y_n) = (x_1y_1, x_2y_2, \dots, x_2y_n).$$

(1) Show that A is a unital Banach algebra and find its unit.
(2) Show that the unit has norm one if and only if $p = \infty$.
(3) Apply on A the renorming from Proposition VIII.3 and show, that the new norm is just $\|\cdot\|_\infty$.

Problem 3. Let $A = \ell^p(\Gamma)$, where $p \in [1, \infty)$ and Γ is an infinite set. Equip A with the pointwise multiplication. Show that A is a Banach algebra with no unit.

Problem 4. Let M_n be the algebra of complex $n \times n$-matrices equipped with the matrix multiplication. Recall that any $n \times n$-matrix represents a linear mapping $\mathbb{C}^n \to \mathbb{C}^n$ and that the matrix multiplication corresponds to composition of linear mappings.

(1) Fix $p \in [1, \infty]$ and equip M_n with the operator norm coming from $L((\mathbb{C}^n, \|\cdot\|_p))$. Show that M_n is then a unital Banach algebra and that the unit has norm one.
(2) Show that for $p_1 \neq p_2$ the two norms defined in (1) are equivalent but different whenever $n \geq 2$.

Problem 5. Let M_n be the algebra of complex $n \times n$-matrices equipped with the matrix multiplication. Equip M_n with the norm
$$\|(a_{ij})_{i,j=1,\dots,n}\| = \sum_{i,j=1}^{n} |a_{ij}|\,.$$
Show that M_n equipped with this norm is a unital Banach algebra and its unit has norm greater than 1 (whenever $n \geq 2$).

Problem 6. Let X be any nontrivial Banach space. Define on X the trivial multiplication, i.e., $x \cdot y = \boldsymbol{o}$ for $x, y \in X$.

(1) Show that X is a Banach algebra with no unit.
(2) Describe the unital algebra X^+.
(3) Find a subalgebra of the matrix algebra M_n (where $n \geq 2$) isomorphic with such a trivial algebra.

Problem 7. Let $A_1, \dots, A_n$ be Banach algebras and let $p \in [1, \infty]$. Consider the vector space $A = A_1 \times A_2 \times \cdots \times A_n$, where the norm and multiplication are defined by

$$\|(a_1, \dots, a_n)\| = \|(\|a_1\|, \dots, \|a_n\|)\|_p,$$
$$(a_1, \dots, a_n) \cdot (b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n).$$

(1) Show that A is a Banach algebra.
(2) Show that A is unital if and only if $A_1, \dots, A_n$ are unital.

Problem 8. Let K be a compact Hausdorff space and let A be a Banach algebra. Let $\mathcal{C}(K, A)$ be the vector space of all the continuous mappings $f : K \to A$. Equip $\mathcal{C}(K, A)$ with the norm and with the multiplication given by

$$\|f\| = \sup\{\|f(t)\|; t \in K\}, \qquad f \in \mathcal{C}(K, A),$$
$$(f \cdot g)(t) = f(t) \cdot g(t), \quad t \in K, \quad f, g \in \mathcal{C}(K, A).$$

(1) Show that $\mathcal{C}(K, A)$ is a Banach algebra.
(2) Show that $\mathcal{C}(K, A)$ is unital if and only if A is unital and find the unit.
(3) Show that $\mathcal{C}(K, A)$ is commutative if and only if A is commutative.

Problem 9. Let $(G, +)$ be a commutative group. Equip the Banach space $\ell^1(G)$ with the multiplication $*$ defined by

$$(f * g)(x) = \sum_{y \in G} f(y) g(x - y), \qquad f, g \in \ell^1(G).$$

Show that $\ell^1(G)$ is then a unital commutative Banach algebra and find its unit.

Problem 10. Let $(G, \cdot)$ be a non-commutative group. Equip the Banach space $\ell^1(G)$ with the multiplication $*$ defined by

$$(f * g)(x) = \sum_{y \in G} f(y) g(y^{-1} x), \qquad f, g \in \ell^1(G).$$

Show that $\ell^1(G)$ is then a unital non-commutative Banach algebra and find its unit.

Problem 11. Let $(G, +)$ be a commutative compact topological group. (I.e., $(G, +)$ is a commutative group equipped with a Hausdorff topology in which the operations $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous, which is moreover compact in this topology.) Let $\mathcal{M}(G)$ be the space of all the complex Radon measures on G, equipped with the total variation norm and with the multiplication $*$ defined by

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in G \times G; x + y \in A\}),$$

where $\mu \times \nu$ denotes the respective product measure. Show that $\mathcal{M}(G)$ is then a unital commutative Banach algebra and find its unit.

Problem 12. Let $(G, \cdot)$ be a non-commutative compact topological group. (I.e., $(G, \cdot)$ is a non-commutative group equipped with a Hausdorff topology in which the operations $(x, y) \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are continuous, which is moreover compact in this topology.) Let $\mathcal{M}(G)$ be the space of all the complex Radon measures on G, equipped with the total variation norm and with the multiplication $*$ defined by

$$(\mu * \nu)(A) = (\mu \times \nu)(\{(x, y) \in G \times G; x \cdot y \in A\}),$$

where $\mu \times \nu$ denotes the respective product measure. Show that $\mathcal{M}(G)$ is then a unital non-commutative Banach algebra and find its unit.

Problem 13. Let T be a non-compact locally compact space and $A = \mathcal{C}_0(T)$. Let $B = \operatorname{span}(A \cup \{1\})$ as a subalgebra of $\ell^\infty(T)$. Show that B is (algebraically) isomorphic to A^+, but not isometric.

Problem 14. Let X be an infinite-dimensional Banach space and let $A = K(X)$ be the Banach algebra of compact operators on X. Let $B = \operatorname{span}(A \cup \{I\})$ as a subalgebra of $L(X)$. Show that B is (algebraically) isomorphic to A^+, but not isometric.

Problem 15. Show that in the matrix algebra M_n an element has a right inverse if and only if it has a left inverse.

Problem 16. Let $A = L(\ell^2)$. Define two operators $S, T \in A$ by

$$S(x_1, x_2, \dots) = (x_2, x_3, \dots) \qquad \text{and} \qquad T(x_1, x_2, \dots) = (0, x_1, x_2, \dots).$$

(1) Show that S and T are not invertible.
(2) Show that S has a right inverse and describe all its right inverses.
(3) Show that T has a left inverse and describe all its left inverses.

Problem 17. Let $G = (\mathbb{Z}_n, +)$ where $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ equipped with the addition modulo n. Let $A = \ell^1(G)$ be the Banach algebra described in Problem 9.

(1) Represent A as a subalgebra of the matrix algebra M_n (with an appropriate norm).
(2) For $n = 2$ and $n = 3$ explicitly characterize invertible elements in A.

Problem 18. Let A be a Banach algebra. Define on A a new multiplication $\odot$ by

$$x \odot y = y \cdot x, \quad x, y \in A.$$

(1) Show that $A^{op} = (A, \odot)$ is a Banach algebra.
(2) Show that A^{op} need not be (algebraically) isomorphic to A.
(3) Let X be a reflexive Banach space. Show that $L(X)^{op}$ is isometrically isomorphic to $L(X^*)$.
(4) Let H be a Hilbert space. Show that $L(H)^{op}$ is isometrically isomorphic to $L(H)$.

`Hint:` *(2) Use Problem 1.*

Problems to Section VIII.2 – spectrum and its properties

Problem 19. Let $A = \mathcal{C}(K)$ for a compact Hausdorff space K and let $f \in A$.

(1) Show that $\sigma(f) = f(K)$.
(2) Compute the resolvent function of f.

Problem 20. Let $A = \mathcal{C}_0(T)$ for a noncompact locally compact space T.

(1) Show that $\sigma(f) = f(T) \cup \{0\}$ for each $f \in A$.
(2) Suppose that T is not σ-compact. Show that $\sigma(f) = f(T)$ for each $f \in A$.
(3) In case $T = \mathbb{R}$ find an example of $f \in A$ with $f(T) \subsetneq \sigma(f)$.

Problem 21. Let $A = \ell^1(\mathbb{Z}_n)$ (see Problem 17) and $x \in A$.

(1) Characterize $\sigma(x)$ as the set of eigenvalues of certain matrix.
(2) For $n = 2, 3$ compute $\sigma(x)$ and the resolvent function explicitly.

Problem 22. Let $\mathbb{T} = \{z \in \mathcal{C}; |z| = 1\}$, $A = \mathcal{C}(\mathbb{T})$ and $f(z) = z$ for $z \in \mathbb{T}$. Let B be the unital closed subalgebra of A generated by f, i.e.,

$$B = \overline{\operatorname{span}}\{1, f, f^2, f^3, \dots\}.$$

Compute and compare $\sigma_A(f)$ and $\sigma_B(f)$.

Problem 23. Let A be a unital Banach algebra and let $x \in A$ be such that $x^n = \boldsymbol{o}$ for some $n \in \mathbb{N}$. Determine $\sigma(x)$ and compute the resolvent function.

Problem 24. Let A be a unital Banach algebra and let $x \in A$ be such that $x^2 = x$. Determine $\sigma(x)$ and compute the resolvent function.

`Hint:` *Distinguish three cases: $x = \boldsymbol{o}$, $x = e$ and $x \notin \{\boldsymbol{o}, e\}$. The inverse of $\lambda e - x$ find in the form $\alpha e + \beta x$ for suitable $\alpha, \beta \in \mathbb{C}$.*

Problem 25. Let A be a unital Banach algebra and let $x \in A$ be such that $x^3 = x$. Determine $\sigma(x)$ and compute the resolvent function.

`Hint:` *There are several cases to be distinguished: The case $x^2 = x$ is covered by Problem 24. The case $x^2 = -x$ can be solved similarly as Problem 24. The next case to be solved is $x^2 = e$. Finally, if $x^2 \notin \{e, x, -x\}$, then show that e, x, x^2 are linearly independent and find the inverse of $\lambda e - x$ as a linear combination of e, x, x^2.*

Problem 26. Let $A = \ell^1(\mathbb{Z})$ (cf. Problem 9) and $n \in \mathbb{Z}$, $n \neq 0$. Show that $\sigma(\boldsymbol{e}_n) = \mathbb{T}$ (where $\boldsymbol{e}_n$ is the respective canonical vector) and that

$$R(\lambda, \boldsymbol{e}_n) = \begin{cases} \sum_{k=0}^{\infty} \frac{\boldsymbol{e}_{kn}}{\lambda^{k+1}}, & |\lambda| > 1, \\ \sum_{k=1}^{\infty} -\lambda^k \boldsymbol{e}_{-kn}, & |\lambda| < 1. \end{cases}$$

`Hint:` *This can be proved directly by solving the equation $(\lambda \boldsymbol{e}_0 - \boldsymbol{e}_n) * f = \boldsymbol{e}_0$. One can also use the formula from Proposition VIII.8(v) and its modifications.*

Problems to Section VIII.3 – holomorphic functional calculus

Problem 27. Let A be a unital Banach algebra and let f be an entire function. Let

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n, \qquad \lambda \in \mathbb{C},$$

be its Taylor expansion. Show that for each $x \in A$ we have

$$\tilde{f}(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Problem 28. Let $A = M_n$ and $D \in A$ be a diagonal matrix, with values $d_1, \dots, d_n$ on the diagonal.

(1) Show that $\sigma(D) = \{d_1, \dots, d_n\}$ and compute the resolvent function.
(2) Let f be a function holomorphic on a neighborhood of $\sigma(D)$. Show that $\tilde{f}(D)$ is the diagonal matrix with values $f(d_1), \dots, f(d_n)$ on the diagonal.
(3) Deduce that in this case the value of $\tilde{f}(D)$ depends only on $f|_{\sigma(D)}$.

Problem 29. Let $A = M_n$ where $n \geq 2$ and let $J \in A$ be a Jordan cell, with the value z on the diagonal, i.e.,

$$J = \begin{pmatrix} z & 1 & 0 & \dots & 0 & 0 \\ 0 & z & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z & 1 \\ 0 & 0 & 0 & \dots & 0 & z \end{pmatrix}.$$

(1) Show that $\sigma(J) = \{z\}$.
(2) Show that

$$(\lambda I - J)^{-1} = \begin{pmatrix} \frac{1}{\lambda - z} & \frac{1}{(\lambda-z)^2} & \dots & \frac{1}{(\lambda-z)^n} \\ 0 & \frac{1}{\lambda-z} & \dots & \frac{1}{(\lambda-z)^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\lambda-z} \end{pmatrix} \quad \text{for } \lambda \in \mathbb{C} \setminus \{z\}.$$

(3) Let f be a function holomorphic on a neigborhood of z. Show that

$$\tilde{f}(J) = \begin{pmatrix} f(z) & f'(z) & \frac{f''(z)}{2} & \dots & \frac{f^{(n-1)}(z)}{(n-1)!} \\ 0 & f(z) & f'(z) & \dots & \frac{f^{(n-2)}(z)}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f(z) \end{pmatrix}$$

(4) Deduce that in this case the value of $\tilde{f}(J)$ is not determined by $f|_{\sigma(J)}$.

Problem 30. Let $A = M_n$ and $E \in A$ be an arbitrary matrix. Let f be a function holomorphic on a neighborhood of $\sigma(E)$.

(1) Express $\tilde{f}(E)$ using the Jordan canonical form of E.
(2) Characterize those matrices E for which $\tilde{f}(E)$ is determined by $f|_{\sigma(E)}$.

Problem 31. Let $A = \ell^1(\mathbb{Z}_2)$ or $A = \ell_1(\mathbb{Z}_3)$. For $x \in A$ and f holomorphic on a neighborhood of $\sigma(x)$ compute the value of $\tilde{f}(x)$.

Problem 32. Let A be a unital Banach algebra and $x \in A$ be an element satisfying one of the following conditions:

(1) $x^n = 0$ for some $n \in \mathbb{N}$;
(2) $x^2 = x$;
(3) $x^2 = -x$;
(4) $x^2 = e$;
(5) $x^3 = x$, but none of the conditions (2)–(4) holds.

Let f be a function holomorphic on a neighborhood of $\sigma(x)$. Compute $\tilde{f}(x)$. In which cases it is determined by $f|_{\sigma(x)}$?

Problem 33. Let $A = \mathcal{C}(K)$, let $g \in A$ and let F be a function holomorphic on a neigborhood of $\sigma(g) = g(K)$. Show that $\tilde{F}(g) = F \circ g$.

Problem 34. Let $A = \ell^1(\mathbb{Z})$ (cf. Problem 9) and $n \in \mathbb{Z}$, $n \neq 0$. By Problem 26 we know that $\sigma(\boldsymbol{e}_n) = \mathbb{T}$. Let g be a function holomorphic on a neighborhood of $\mathbb{T}$. Show that

$$\tilde{g}(\boldsymbol{e}_n) = \sum_{k \in \mathbb{Z}} a_k e_{kn},$$

where $(a_k)_{k \in \mathbb{Z}}$ are the coeficients of the Laurent expansion of g in a neighborhood of $\mathbb{T}$.

`Hint:` *One can use either the definitions and the formula from Problem 26, or one can prove an analogue of the statement in Problem 27 for Laurent series.*