Let E be an abstract spectral measure

① $\|E_{x,y}\| \le \|x\|\cdot\|y\|$

$\lceil$ $\|E_{x,y}\| = |E_{x,y}|(C) = \sup\left\{\sum_{j=1}^{n} |E_{x,y}(A_j)| \; ; \; A_1,\dots,A_n \in \mathcal{A} \text{ disjoint}\right\}$

Let $A_1,\dots,A_n \in \mathcal{A}$ be disjoint. Then

$$\sum_{j=1}^{n} |E_{x,y}(A_j)| = \sum_{j=1}^{n} |\langle E(A_j)x,y\rangle| = \sum_{j=1}^{n} \alpha_j \cdot \langle E(A_j)x,y\rangle =$$

$\alpha_j \in \mathbb{C},\ |\alpha_j|=1$ suitable

$$= \sum_{j=1}^{n} \langle \alpha_j \cdot E(A_j)x,y\rangle = \left\langle \sum_{j=1}^{n} \alpha_j \cdot E(A_j)x,y \right\rangle \le$$

$$\le \left\| \sum_{j=1}^{n} \alpha_j \cdot E(A_j)x \right\| \|y\| = \left\| \sum_{j=1}^{n} E(A_j)x \right\| \|y\|$$

A_j disjoint $\Rightarrow E(A_j) \perp E(A_k),\ j \ne k$
$|\alpha_j| = 1$

A_j disjoint

$$= \left\| E\left(\bigcup_{j=1}^{n} A_j\right)x \right\| \|y\| \le \|x\|\cdot\|y\|$$ $\rfloor$

② Let f be a bdd $\mathcal{A}$-measurable function.
Then there is a unique $T \in \mathcal{L}(H)$ s.t.

$$\langle Tx,y\rangle = \int f \, dE_{x,y}, \quad x,y \in H$$

Moreover, $\|T\| \le \|f\|_\infty$. Notation: $T =: \int f \, dE$

$\lceil$ $B(x,y) = \int f \, dE_{x,y}$

Well defined: f bdd $E_{x,y}$-measurable, $E_{x,y}$ finite measure
$x \mapsto B(x,y)$ is linear, $y \mapsto B(x,y)$ is conjugate linear
by the definition of $E_{x,y}$ ... $E_{x,y}(A) = \langle E(A)x,y\rangle$
$x,y \in B_H$: $|B(x,y)| \le \int |f| \, d|E_{x,y}| \le \|f\|_\infty \cdot \|E_{x,y}\| \le$
$\le \|f\|_\infty \cdot \|x\|\,\|y\| \le \|f\|_\infty$. It remains to use Lax-Milgram lemma

③ $\mathcal{N} := \{A \in \mathcal{A},\ E(A)=0\} = \{A \in \mathcal{A};\ \forall x \in H\ \ E_{x,x}(A) = 0\}$

If $f = g$ except on a set from $\mathcal{N}$ $\Rightarrow$ $\int f\,dE = \int g\,dE$

⌈obvious⌋

Define now $L^\infty(E)$ similarly as $L^\infty(E_T)$

Then $\Psi: L^\infty(E) \to L(H)$, $\Psi(f) = \int f\,dE$ is well defined

Moreover, Ψ is linear and $\|\Psi\| \le 1$

④ $\Psi(f)^* = \Psi(\overline{f})$

$\langle \Psi(f)^* x, x\rangle = \langle x, \Psi(f)x\rangle = \overline{\langle \Psi(f)x, x\rangle} = \overline{\int f\,dE_{x,x}} =$

$= \int \overline{f}\,dE_{x,x} = \langle \Psi(\overline{f})x, x\rangle$, use Prop. 4 (c)

↑

$E_{x,x} \ge 0$ ($E_{x,x}(A) = \langle E(A)x, x\rangle \ge 0$ as $E(A)$ is an OG projection)

⑤ $\Psi(\chi_A) = E(A)$, $A \in \mathcal{A}$

⌈$\langle \Psi(\chi_A)x, y\rangle = \int \chi_A\,dE_{x,y} = E_{x,y}(A) = \langle E(A)x, y\rangle$⌋

⑥ $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g)$

⌈ • $f = \chi_A$, $g = \chi_B$ $\Rightarrow$ $\Psi(fg) = \Psi(\chi_{A \cap B}) = E(A \cap B) =$

$= E(A)E(B) = \Psi(f)\,\Psi(g)$

• f given $\Rightarrow$ $\{g;\ \Psi(f \cdot g) = \Psi(f) \cdot \Psi(g)\}$ is a linear subspace

$\{g;\ \Psi(g \cdot f) = \Psi(g) \cdot \Psi(f)\}$ as well

So $\Psi(f \cdot g) = \Psi(f) \cdot \Psi(g)$ if f, g are simple $\mathcal{A}$-measurable

- g simple, f general
  $x, y \in H$
  $\exists\ (f_n)$ simple Borel meas, $\|f_n\|_\infty \le \|f\|_\infty$
  s.t. $f_n \to f$ $\quad |E_{x,y}| + |E_{\Psi(g)x,y}|$ – a.e.

Then $\langle \Psi(f)\Psi(g)x, y\rangle = \int f \, dE_{\Psi(g)x,y} = \lim_n \int f_n \, dE_{\Psi(g)x,y}$

$= \lim_n \langle \Psi(f_n)\Psi(g)x, y\rangle = \lim_n \langle \Psi(f_n g)x, y\rangle =$

$= \lim_n \int f_n g \, dE_{x,y} = \int fg \, dE_{x,y} = \langle \Psi(fg)x, y\rangle$

- g, f general
  $x, y \in H$
  $\exists\ (g_n)$ simple Borel meas. $\|g_n\|_\infty \le \|g\|_\infty$
  s.t. $g_n \to g$ $\quad |E_{x,y}| + |E_{x, \Psi(f)^* y}|$ – a.e.

Then $\langle \Psi(f)\Psi(g)x, y\rangle = \langle \Psi(g)x, \Psi(f)^* y\rangle =$

$= \int g \, dE_{x, \Psi(f)^* y} = \lim \int g_n \, dE_{x, \Psi(f)^* y} =$

$= \lim_n \langle \Psi(g_n)x, \Psi(f)^* y\rangle = \lim_n \langle \Psi(f)\Psi(g_n)x, y\rangle =$

$= \lim_n \langle \Psi(f g_n)x, y\rangle = \lim_n \int f g_n \, dE_{x,y} = \int fg \, dE_{x,y} =$

$= \langle \Psi(fg)x, y\rangle$ ┘

(7) Summarizing (3) – (6) : Ψ is a $*$-homomorphism

$L^\infty(E) \to \mathcal{L}(H)$

⑧ $\|\psi(f)x\| = \left(\int |f|^2 dE_{x,x}\right)^{1/2} = \|f\|_{L^2(E_{x,x})}$

$\lceil\ \|\psi(f)x\|^2 = \langle \psi(f)x, \psi(f)x\rangle = \langle \psi(f)^* \psi(f)x, x\rangle =$

$= \langle \psi(\bar{f}\cdot f)x, x\rangle = \int |f|^2 dE_{x,x}$ $\rfloor$

⑨ ψ is one-to-one, hence it is an isometry

$\lceil\ \psi(f) = 0 \Rightarrow \forall x \quad \psi(f)x = 0 \overset{⑧}{\Rightarrow} \forall x \ f = 0 \ E_{x,x}$-a.e.

$\Rightarrow f = 0$ except on a set from $\mathcal{N}$ $\Rightarrow f = 0$ in $L^\infty(E)$ $\rfloor$

⑩ $\psi(f)$ is always normal, $\psi(f)$ self-adjoint $\Leftrightarrow$ f real-valued (except on a set from $\mathcal{N}$)

$\psi(f) \geq 0 \Leftrightarrow f \geq 0$ (except on a set from $\mathcal{N}$)

$\lceil\ \psi(f)^* \psi(f) = \psi(\bar{f} f) = \psi(f \cdot \bar{f}) = \psi(f)\cdot\psi(f)^*$

f real valued $\Rightarrow \langle \psi(f)x, x\rangle = \int f \, dE_{x,x} \in \mathbb{R}$ for $x \in H$

$f \geq 0 \Rightarrow \langle \psi(f)x, x\rangle \geq 0$ $\left(= \int f \, dE_{x,x}\right) \geq 0$ for $x \in H$

[[as $E_{x,x} \geq 0$]]

Conversely: $\psi(f)$ self-adjoint $\Rightarrow \forall x \ \langle \psi(f)x, x\rangle \in \mathbb{R}$

$\langle \psi(f)x, x\rangle = \int f \, dE_{x,x} = \underbrace{\int \mathrm{Re} f \, dE_{x,x}}_{\in \mathbb{R}} + i \underbrace{\int \mathrm{Im} f \, dE_{x,x}}_{\in \mathbb{R}}$

$\Rightarrow \mathrm{Im} f = 0 \ E_{x,x}$-a.e. This holds for all $x \Rightarrow \mathrm{Im} f = 0$ except on a set from $\mathcal{N}$

$\psi(f) \geq 0 \Rightarrow f$ real valued (by above), $f = f^+ - f^-$. ~~Suppose $f^- \neq 0$~~

Then

$\forall x \in H$:

$$0 \leq \langle \psi(f) \cdot \psi(f^-)x, \psi(f^-)x \rangle \overset{\psi(f^-) \text{ self-adjoint}}{=} \langle \psi(f^-)\psi(f)\psi(f^-)x, x \rangle$$

$$= \langle \psi(f^- \circ f \circ f^-)x, x \rangle = \langle \psi(-(f^-)^3)x, x \rangle \leq 0$$

So, $\psi((f^-)^3) = 0$, thus $(f^-)^3 = 0$, so $f^- = 0$

(except on a set from $\mathcal{N}$ ⌋

(11) $g \in \mathcal{C}(\sigma(\psi(f))) = \mathcal{C}(\text{ess-rng} f) \Rightarrow \psi(g \circ f) = \tilde{g}(\psi(f))$

$T := \psi(f)$, f fixed

consider $A = \{ g \in \mathcal{C}(\sigma(\psi(f))) ; \psi(g \circ f) = \tilde{g}(\psi(f)) \}$

Then
- A is a linear subspace [clear]
- $g_1, g_2 \in A \Rightarrow g_1 \cdot g_2 \in A$

⌈ $\psi((g_1 g_2) \circ f) = \psi((g_1 \circ f)(g_2 \circ f)) = \psi(g_1 \circ f)\psi(g_2 \circ f)$

$= \tilde{g_1}(\psi(f))\tilde{g_2}(\psi(f)) = \widetilde{g_1 g_2}(\psi(f))$ ⌋

- $g \in A \Rightarrow \bar{g} \in A$

⌈ $\psi(\bar{g} \circ f) = \psi(\overline{g \circ f}) = \psi(g \circ f)^* = \tilde{g}(\psi(f))^* =$

$= \widetilde{\bar{g}}(\psi(f))$ ⌋

- $1 \in A$: $\psi(1 \circ f) = \psi(1) = I = \tilde{1}(\psi(f))$

- $id \in A$: $\psi(id \circ f) = \psi(f) = \tilde{id}(\psi(f))$

So, A separates points of $\sigma(\psi(f))$

Thus $A = \mathcal{C}(\sigma(\psi(f)))$ by S-W thm, as A is also closed

by continuity of $g \mapsto \psi(g \circ f) - \tilde{g}(\psi(f))$ ⌋

(12) $T = \int f dE \Rightarrow E_T(A) = E(f^{-1}(A))$

$\ulcorner$ Denote $E'(A) = E(f^{-1}(A))$. Then

For $x, y \in H$, $g \in C(\sigma(\varphi(f)))$

$\langle \tilde{g}(T)x, y\rangle = \langle \tilde{g}(\varphi(f))x, y\rangle = \langle \varphi(g \circ f)x, y\rangle =$

$= \int g \circ f \, dE_{x,y} = \int g \, d\, f(E_{x,y}) = \int g \, d E'_{x,y}$ $\lrcorner$

(13) T normal operator $\Rightarrow \exists!\, E$ abstract spectral measure s.t. $\int id \, dE = T$

Moreover, $E = E_T$

$\ulcorner T = \int id \, dE \overset{(12)}{\Rightarrow} E_T = E$.

$\langle (\int id \, dE_T)x, y\rangle = \int id \, dE^{(T)}_{x,y} = \tilde{id}(T) = T$ $\lrcorner$