The mapping E_T is an abstract spectral measure, it is compactly supported

<u>Recall</u> $\mathcal{A} = \{A \subset \sigma(T) ; A \text{ is } E_{x,y}\text{-measurable for each } x,y \in H\}$

$A \in \mathcal{A} \Rightarrow E_T(A) = \tilde{\chi}_A(T)$

Set $\tilde{\mathcal{A}} = \{A \subset \mathbb{C} ; A \cap \sigma(T) \in \mathcal{A}\}$

$E_T(A) = E_T(A \cap \sigma(T))$, $\quad A \in \tilde{\mathcal{A}}$

<u>Then:</u> (i) $\tilde{\mathcal{A}}$ is a σ-algebra of subsets of $\mathbb{C}$ ~~set~~. containing all the Borel sets [clear]

(ii) $\forall A \in \tilde{\mathcal{A}}$ $E_T(A)$ is an orthogonal projection

$$\ulcorner E_T(A) = \widetilde{\chi_{A \cap \sigma(T)}}(T)$$

- $\chi_{A\cap\sigma(T)}$ real valued $\Rightarrow E_T(A)$ self-adjoint
- $(\chi_{A\cap\sigma(T)})^2 = \chi_{A\cap\sigma(T)} \Rightarrow (E_T(A))^2 =$

$$= (\widetilde{\chi_{A\cap\sigma(T)}}(A))^2 = \widetilde{(\chi_{A\cap\sigma(T)})^2}(A) = E_T(A)$$

So, $E_T(A)$ is a self-adjoint projection, hence it is an OG projection (Prop. 6) $\lrcorner$

(iii) $E_T(\emptyset) = \tilde{\chi}_\emptyset(T) = \tilde{0}(T) = 0$

$E_T(\mathbb{C}) = \tilde{\chi}_{\sigma(T)}(T) = \tilde{1}(T) = I$

(IV) $A \in \tilde{\mathcal{A}}$, $E_T(A) = 0 \Rightarrow \forall B \subset A : B \in \tilde{\mathcal{A}}$ & $E_T(B) = 0$

$A \in \tilde{\mathcal{A}}$, $E_T(A) = 0 \Rightarrow \forall x \in H : 0 = \langle E_T(A)x, x\rangle =$

$= \langle \tilde{\chi}_{A\cap\sigma(T)}(T)x, x\rangle = \int \chi_{A\cap\sigma(T)} \, dE_{x,x} = E_{x,x}(A\cap\sigma(T))$

Hence for each $B \subset A$ $\quad B \cap \sigma(T) \in \mathcal{R}$ & $E_{x,t}(B \cap \sigma(T)) = 0$ for $t \in H$

Thus $B \in \tilde{\mathcal{A}}$ and $\langle E_T(B)t, t\rangle = 0$ for $t \in H$, so $E_T(B) = 0$ ⌋

(v) $E_T(A \cap B) = \widetilde{\chi_{A \cap B \cap \sigma(T)}}(T) = \widetilde{\chi_{A \cap \sigma(T)} \cdot \chi_{B \cap \sigma(T)}}(T) =$

$= \widetilde{\chi_{A \cap \sigma(T)}}(T) \cdot \widetilde{\chi_{B \cap \sigma(T)}}(T) = E_T(A) E_T(B)$

(vi) $A \cap B = \emptyset \Rightarrow E_T(A \cup B) = \widetilde{\chi_{(A \cup B) \cap \sigma(T)}}(T) =$

$= \left(\widetilde{\chi_{A \cap \sigma(T)} + \chi_{B \cap \sigma(T)}}\right)(T) = \widetilde{\chi_{A \cap \sigma(T)}}(T) + \widetilde{\chi_{B \cap \sigma(T)}}(T) = E_T(A) + E_T(B)$

(vii) $\langle E_T(A)x, y\rangle = \langle \widetilde{\chi_{A \cap \sigma(T)}}(T)x, y\rangle = \int \chi_{A \cap \sigma(T)} \, d\tilde{e}_{x,y} =$

$= E_{x,y}(A \cap \sigma(T))$, so it is a complex Borel measure as $E_{x,y}$ is such.

• E_T is compactly supported as $E_T(\mathbb{C} \setminus \sigma(T)) = 0$