# Invertibility and spectrum in $A$ and $A^+$

① $a \in A \Rightarrow (a,0)$ is not invertible in $A^+$

$\Gamma\ (a,0)\cdot(b,\lambda) = (ab + \lambda a, 0) \neq (0,1)$ ⅃

② $A$ has no unit $\Rightarrow \forall x \in A \quad \sigma(x) \ni 0$

$\Gamma$ By definition $\sigma(x) = \sigma_{A^+}((x,0)) \ni 0$, as $(x,0)$ is not invertible by ① ⅃

③ $A$ has a unit $e \Rightarrow (a,\lambda) \in A^+$ is invertible iff $\lambda \neq 0$ & either $a = 0$ or $a + \lambda e$ is invertible

$\Gamma$ Recall $(a,\lambda)(b,\mu) = (ab + \lambda b + \mu a, \lambda\mu)$

$\Rightarrow$ Suppose that $(a,\lambda)$ is invertible

- $\exists (b,\mu)$ s.t. $(a,\lambda)\cdot(b,\mu) = (0,1)$
  in particular $\lambda\mu = 1$, hence $\lambda \neq 0$

Suppose moreover that $a \neq 0$

- $\exists (b,\mu)$ s.t. $(a,\lambda)\cdot(b,\mu) = (e,0)$

  Hence $ab + \lambda b + \mu a = e$

  $\lambda\mu = 0$

  Since $\lambda \neq 0$, necessarily $\mu = 0$.

  Thus $\underbrace{ab + \lambda b}_{(a+\lambda e)b} = e$

  $\Rightarrow a + \lambda e$ has a right inverse

- Similarly ... $\exists (b,\mu)$ s.t. $(b,\mu)(a,\lambda) = (e,0)$

  Then $\underbrace{ba + \lambda b}_{b(a+\lambda e)} = e \Rightarrow a + \lambda e$ has a left inverse

So, $a + \lambda e$ is invertible

$\Leftarrow$: • $\lambda \neq 0, a = 0$ ... $(0, \lambda)^{-1} = (0, \frac{1}{\lambda})$

• $\lambda \neq 0$, $a + \lambda e$ invertible. Let us look for $(b, \mu)$ with
$(a, \lambda)(b, \mu) = (0, 1)$
then $\lambda \mu = 1 \Rightarrow \mu = \frac{1}{\lambda}$

and $0 = ab + \lambda b + \mu a = ab + \lambda b + \frac{1}{\lambda} a$

$$0 = (a + \lambda e) b + \frac{1}{\lambda} a \Rightarrow b = \frac{1}{\lambda}(a + \lambda e)^{-1} a$$

Then, indeed, $\left(\frac{1}{\lambda}(a+\lambda e)^{-1} a, \frac{1}{\lambda}\right)$ is a right inverse
Similarly, $\left(\frac{1}{\lambda} a \cdot (a + \lambda e)^{-1}, \frac{1}{\lambda}\right)$ is a left inverse
so $(a, \lambda)$ is invertible.

(4) $A$ has a unit $e$ $\Rightarrow \sigma_{A^+}(a, 0) = \sigma_A(a) \cup \{0\}$

More generally:
$\lambda \in \sigma_{A^+}(a, \mu) \Leftrightarrow (-a, \lambda - \mu)$ is not invertible

$\overset{(3)}{\Leftrightarrow}$ $\lambda - \mu = 0$ or ~~is not~~ $(\lambda - \mu)e - a$ is not invertible and ~~not zero~~ $a \neq 0$

$\Leftrightarrow \lambda = \mu$ or $(\lambda - \mu \in \sigma_A(a)$ & ~~$a \neq (\lambda-\mu)e$~~ $a \neq 0)$

So, if $a \neq 0$, then $\sigma_{A^+}(a, \mu) = \{\mu\} \cup (\mu + \sigma_A(a))$
$a = 0$: $\sigma_{A^+}(0, \mu) = \{\mu\}$

So, in both cases $\sigma_{A^+}(a, \mu) = \{\mu\} \cup (\mu + \sigma_A(a))$

In particular, $\sigma_{A^+}(a, 0) = \{0\} \cup \sigma_A(a)$.