Weak Stegall spaces

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Remark on references. The unspecified references, and the spaces K_B as well, are from the paper O.Kalenda, Stegall compact spaces which are not fragmentable, Topol. Appl. 96 (1999), no.2, 121–132.

Proposition W1. *Let X be a topological space. Then the following assertions are equivalent.*

(i) Any minimal usco mapping of any complete metric space M into X is singlevalued at least at one point of M.

(ii) Any minimal usco mapping of any complete metric space M into X is singlevalued at points of a dense subset of M.

(iii) Any minimal usco mapping of any complete metric space M into X is singlevalued at points of a second category subset of M.

(iv) Any minimal usco mapping of any complete metric space M into X is singlevalued at points of a dense Baire subspace of M.

Proof. The implications $(iv) \Rightarrow (iii) \Rightarrow (i)$ and $(iv) \Rightarrow (ii) \Rightarrow (i)$ are obvious. It remains to prove $(i) \Rightarrow (iv)$. Let M be a complete metric space, $\varphi : M \to X$ a minimal usco mapping such that $A = \{m \in M \mid \varphi(x)$ is not a singleton$\}$ is not a dense Baire subspace of M. Then there is $U \subset M$ nonempty open such that $U \cap A$ is meager in U, and a dense G_δ subset G of U such that $G \cap A = \emptyset$. We apply twice Lemma 2 to get that $\varphi \upharpoonright G$ is a minimal usco mapping. Moreover, G is completely metrizable, and $\varphi \upharpoonright G$ is not singlevalued at any point of G, which completes the proof. $\square$

A space X satisfying one of the equivalent conditions of the above proposition we will call a *weakly Stegall* space, or we will write $X \in w - \mathcal{S}$.

Proposition W2. *(a) Let $X \in w\text{-}\mathcal{S}$ and $f : Y \to X$ be continuous one-to-one. Then $Y \in w\text{-}\mathcal{S}$.*

(b) If $X = \bigcup_{n\in\mathbb{N}} X_n$ with each X_n closed in X, and if $X_n \in w\text{-}\mathcal{S}$ for every n, then $X \in w\text{-}\mathcal{S}$.

(c) If $X \in w\text{-}\mathcal{S}$ and Y is a perfect image of X then $Y \in w\text{-}\mathcal{S}$. In particular, continuous image af a compact space lying in $w\text{-}\mathcal{S}$ lies in $w\text{-}\mathcal{S}$ too.

(d) If $X \in w\text{-}\mathcal{S}$ and $Y \in \mathcal{S}$ then $X \times Y \in w\text{-}\mathcal{S}$.

Proof. (a) If M is a complete metric space and $\varphi : M \to Y$ is a minimal usco, then, by Lemma 1, $f \circ \varphi$ is also a minimal usco. Since $X \in w\text{-}\mathcal{S}$, there is $m \in M$ such that $f(\varphi(m))$ is a singleton. Now, since f is one-to-one, $\varphi(m)$ is a singleton too.

(b) Let M be a complete metric space and $\varphi : M \to X$ a minimal usco. Put $M_n = \varphi^{-1}(X_n)$. Then M_n is a sequence of closed sets covering M, hence there is some n such that M_n has nonempty interior in M. Let $U \subset M_n$ be nonempty open. By Lemma 1(c) we get $\varphi(U) \subset X_n$. By Lemma 2 the restriction $\varphi \upharpoonright U$ is minimal usco. Since $X_n \in w\text{-}\mathcal{S}$, there is $m \in U$ such that $\varphi(m)$ is a singleton.

(c) Let $f : X \to Y$ be a perfect mapping of X onto Y. Then f^{-1} is an usco mapping. Let $\varphi : M \to Y$ be a minimal usco, where M is a complete metric space. Then $f^{-1} \circ \varphi$ is usco. Let $\psi \subset f^{-1} \circ \varphi$ be a minimal usco. Then there is $m \in M$ such that $\psi(m)$ is a singleton. Clearly we have $f \circ \psi \subset \varphi$, hence, by minimality of φ, $f \circ \psi = \varphi$. Therefore $\varphi(m) = f(\psi(m))$ is a singleton.

(d) Let M be a complete metric space and $\varphi : M \to X \times Y$ be a minimal usco. Then $\pi_X \circ \varphi$ is a minimal usco $M \to X$, so there is $A \subset M$ of second category such that $\pi_X \circ \varphi$ is singlevalued at all points of A. Similarly $\pi_Y \circ \varphi$ is singlevalued at points of a residual set $B \subset M$ (since $Y \in \mathcal{S}$). Then φ is singlevalued at points of $A \cap B$, which is a nonempty set. $\square$

Lemma W1. *Let M be a complete metric space and $f : M \to X$ a continuous map such that for every $U \subset M$ open $f(U)$ has no isolated points. Then there is a nonempty compact perfect set $P \subset M$ such that $f \restriction P$ is one-to-one.*

Proof. Let ρ be a complete metric on M such that $\rho \leq 1$. We can construct by induction nonempty open sets $U_s \subset M$ indexed by finite sequences of 0 and 1 satisfying

(i) $\overline{U_{s^\frown 0}} \cup \overline{U_{s^\frown 1}} \subset U_s$,

(ii) $f\left(\overline{U_{s^\frown 0}}\right) \cap f\left(\overline{U_{s^\frown 1}}\right) = \emptyset$,

(iii) $\operatorname{diam} U_s \leq 2^{-|s|}$.

Put $U_\emptyset = M$. If we have costructed U_s then by the assumption on f we get that $f(U_s)$ has no isolated points and hence we can choose two distinct points $x_0, x_1 \in f(U_s)$. Choose V_0, V_1 two disjoint open neighborhoods of x_0, x_1 and $U_{s^\frown i}$ of sufficiently small diameter such that $\overline{U_{s^\frown i}} \subset U_s \cap f^{-1}(V_i)$ for $i = 0, 1$. This completes the construction.

Now put $K = \bigcup_{\alpha \in 2^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} U_{\alpha \restriction n}$. Then K is a compact perfect set and $f \restriction K$ is one-to-one by the construction. $\square$

Remark. By a similar method one can prove that whenever M is Čech complete and $f : M \to X$ is in the lemma, there is a compact set $K \subset M$ such that $f(K)$ is uncountable.

Proposition W3. *Let $K \subset \mathbb{R}$ be a compact perfect set, $B \subset K^d$ arbitrary. Then $K_B \in w\text{-}\mathcal{S}$ if and only of B does not contain any perfect subset.*

Proof. Let $F : K_B \to K$ be the natural surjection. If B contains a perfect set P then $F^{-1} : P^d \to K_B$ is, by Proposition 6(6), a minimal usco. Moreover, P^d is completely metrizable and F^{-1} is not singlevalued at any point of P^d.

Now suppose that B contains no perfect set. Let M be a complete metric space and $\varphi : M \to K_B$ a minimal usco, nowhere singlevalued. By Proposition 6(5) there is $G \subset M$ dense G_δ such that for $m \in G$ we have $\varphi(m) \subset \{x\} \times \{0, 1\}$ for some $x \in K$. So $\varphi \restriction G$ is a minimal usco (Lemma 2) which is exactly 2-valued. By Proposition 6(6) we get that $F \circ \varphi : G \to B$ satisfies the assumptions of Lemma W1. Hence B contains a perfect set, a contradiction. $\square$

Lemma W2. *Let $\varphi_a : M_a \to X_a$ be an usco mapping for each $a \in A$. Put $M = \prod_{a \in A} M_a$, $X = \prod_{a \in A} X_a$ and let $\varphi : M \to X$ be defined by the formula $\varphi\left((m_a)_{a \in A}\right) = \prod_{a \in A} \varphi_a(m_a)$. Then φ is an usco mapping. Moreover, if each φ_a is minimal so is φ.*

Proof. We denote by π_a the projection of X (or M) onto the a-th coordinate. Similarly for any $F \subset A$ the projection onto $\prod_{a \in F} X_a$ (or $\prod_{a \in F} M_a$) is denoted by π_F.

Clearly the values of φ are compact. Let $m \in M$ and $U \subset X$ be open with $\varphi(m) \subset U$. By the definition of the product topology we get for every $x \in \varphi(m)$ a finite set $F_x \subset A$ and an open set V_x in $\prod_{a \in F_x} X_a$ such that $x \in \pi_{F_x}^{-1}(V_x) \subset U$.

By compactness of $\varphi(m)$ there is $H \subset \varphi(m)$ with $\varphi(x) \subset \bigcup_{x\in H} \pi_{F_x}^{-1}(V_x) \subset U$. Put $F = \bigcup_{x\in H} F_x$. Then there is an open set V in $\prod_{a\in F} X_a$ such that $\bigcup_{x\in H} \pi_{F_x}^{-1}(V_x) = \pi_F^{-1}(V)$. Hence $\varphi(m) \subset \pi_F^{-1}(V) \subset U$. Now, if there is no neighborhood W of m with $\varphi(W) \subset \pi_F^{-1}(V)$ then there is a net $m^\tau \in M$ converging to m and $x^\tau \in \varphi(M^\tau) \setminus \pi_F^{-1}(V)$. Since each φ_a is usco, there is a subnet of x_a^τ converging to some point of $\varphi_a(m_a)$. And since F is finite we can without loss of generality suppose that for each $a \in F$ the net x_a^τ converges to some $x_a \in \varphi_a(m_a)$. So there is τ_0 such that for $\tau \geq \tau_0$ we have $(x_a^\tau)_{a\in F} \in V$, so $x^\tau \in \pi_F^{-1}(V)$, a contradiction. Hence φ is usco.

Next suppose that each φ_a is minimal. Let $U \subset M$ and $W \subset X$ be open with $\varphi(U) \cap W \neq \emptyset$. Again by the definition of product topology there is $F \subset A$ finite and open sets $U_a \subset M_a$ and $W_a \subset X_a$ such that $\bigcap_{a\in F} \pi_a^{-1}(U_a) \subset U$, $\bigcap_{a\in F} \pi_a^{-1}(W_a) \subset W$ and $\varphi\left(\bigcap_{a\in F} \pi_a^{-1}(U_a)\right) \cap \left(\bigcap_{a\in F} \pi_a^{-1}(U_a)\right) \neq \emptyset$. It follows, by definition of φ, that $\varphi_a(U_a) \cap W_a \neq \emptyset$ for every $a \in F$. Since φ is minimal, by Lemma 1, we get a nonempty open $V_a \subset U_a$ with $\varphi_a(V_a) \subset W_a$. So $\varphi\left(\bigcap_{a\in F} \pi_a^{-1}(V_a)\right) \subset \left(\bigcap_{a\in F} \pi_a^{-1}(U_a)\right)$, hence φ is minimal by Lemma 1. $\square$

Example W1. *Let $K = [0,1]$. There is $B \subset (0,1)$ such that $K_B \in w\text{-}\mathcal{S}$ but $K_B \times K_B \notin w\text{-}\mathcal{S}$.*

Proof. By [J.Oxtoby, Measure and category, Springer-Verlag 1971] there is $D \subset \mathbb{R}$ such that neither D nor its complement contain a perfect compact set. Put $B = \left(D \cap \left(0, \frac{1}{2}\right)\right) \cup \left(\frac{1}{2} + \left(\left(0, \frac{1}{2}\right) \setminus D\right)\right)$. Then clearly B contains no perfect compact set, so by Proposition W3 we get that $K_B \in w\text{-}\mathcal{S}$. We will show that the product $K_B \times K_B$ contain a homeomorphic copy of $K_{(0,1)}$ and hence it is not weakly Stegall (by Propositions W2 and W3). Let us define $f : K_{(0,1)} \to K_B \times K_B$ by the formula $f((t,\varepsilon)) = (f_1((t,\varepsilon)), f_2((t,\varepsilon)))$, where

$$f_1((t,\varepsilon)) = \begin{cases} \left(\frac{t}{2}, \varepsilon\right) & \frac{t}{2} \in B \\ \left(\frac{t}{2}, 0\right), & \frac{t}{2} \notin B \end{cases}, \quad f_2((t,\varepsilon)) = \begin{cases} \left(\frac{1}{2} + \frac{t}{2}, 0\right) & \frac{t}{2} \in B \\ \left(\frac{1}{2} + \frac{t}{2}, \varepsilon\right) & \frac{t}{2} \notin B \end{cases}.$$

It is easy to see (by Proposition 6(1)) that f_1 and f_2 are continuous, so f is countinuous too. And it follows easily from the definition of B that f is one-to-one. $\square$