Universal quadratic forms over number fields

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Outline

Quadratic forms

Quadratic form $Q(X_1, \dots, X_r) = a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 + \dots$ with $a_{ij} \in \mathbb{Z}$
Interested in which integers are represented

Rich history!

Example

- Pythagorean triples $X^2 + Y^2 - Z^2 = 0$ (eg. right triangle 3,4,5)
  Babylonia 1800 BC
- Pell equation $X^2 - dY^2 = 1$
  Greece 400 BC, India 600 CE
  Fermat 1650s $\quad 1766319049^2 - 61 \cdot 226153980^2 = 1$
- sum of four squares $X^2 + Y^2 + Z^2 + T^2 =$ any positive integer
  Lagrange 1770

Universal quadratic forms

A quadratic form is *universal* if it represents all positive integers.
Many indefinite forms, eg. $X^2 - Y^2 - dZ^2$ with $4 \nmid d$.

More interesting are positive definite forms.

- No universal positive forms in 3 variables
- Lagrange (1770): $X^2 + Y^2 + Z^2 + T^2$
- Ramanujan, Dickson (1916): classified quaternary universal positive diagonal forms, eg. $X^2 + 2Y^2 + 4Z^2 + dT^2$ with $d \leq 14$

290 Theorem (Bhargava–Hanke 2011)

A positive definite quadratic form over $\mathbb{Z}$ *is universal* $\Longleftrightarrow$ *it represents* $1, 2, 3, \ldots, 290$.

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Number fields

Crucial to study not only $\mathbb{Z}, \mathbb{Q}$, but also *number fields*

$=$ extensions by an algebraic number, eg. $i = \sqrt{-1}, \sqrt[5]{17+\sqrt{2}}, e^{2\pi i/n}, \dots$

Example

- Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$
- $\mathbb{Z}[e^{2\pi i/n}]$ useful for Fermat's Last Theorem $X^n + Y^n = Z^n$

Share many properties with $\mathbb{Z}$

Problem: Unique factorization into product of primes does **not** typically hold

Class number of number field measures the failure of unique factorization

$= 1 \iff$ unique factorization holds

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Quadratic forms over number fields

K = totally real number field (eg. $K = \mathbb{Q}(\sqrt{2})$)
$\mathcal{O}_K$ = ring of integers in K (eg. $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$)

Quadratic form $Q(X_1, \dots, X_r) = a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 + \dots$ with $a_{ij} \in \mathcal{O}_K$ is *universal* if

- it is totally positive definite and
- represents all totally positive elements of $\mathcal{O}_K$

How about sum of squares $X_1^2 + X_2^2 + \cdots + X_r^2$?
Siegel (1945): Universal only for

- $K = \mathbb{Q}$ $r = 4$
- $K = \mathbb{Q}(\sqrt{5})$ $r = 3$

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Universal forms

Questions.

1. Kitaoka's Conjecture: There are only finitely many K with universal *ternary* form
2. How does the minimal number of variables r depend on K?
3. Is there a variant of 290 Theorem?

Previous results.

- Earnest–Khosravani (1997): no ternary universal forms over fields of odd degree
- Chan–Kim–Raghavan (1996): Determined all ternary universal forms over quadratic fields $\mathbb{Q}(\sqrt{D})$
- Kim (1999): 8-ary universal form over each $\mathbb{Q}(\sqrt{n^2-1})$
- Kim-Kim-Park (2021): only finitely many $\mathbb{Q}(\sqrt{D})$ admit 7-ary universal forms

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Why important?

(Why) should we care about universal forms?

For many reasons!

- very classical topic in number theory
- still very modern
  Fields Medal: Bhargava (2014), Venkatesh (2018)

rich connections to

- algebraic, analytic number theory (class numbers, arithmetic statistics, L-functions, modular forms)
- geometry and optimization (best approximations, generalized continued fractions)
- post-quantum cryptography (based on quadratic lattices)

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Real quadratic fields

$D > 1$ squarefree, $D \equiv 2, 3 \pmod 4$
$K = \mathbb{Q}(\sqrt{D})$
$\mathcal{O}_K = \mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} \mid a, b \in \mathbb{Z}\}$

Want to represent *totally positive* elements:

$$\mathcal{O}_K^+ = \{a + b\sqrt{D} \in \mathcal{O}_K \mid a + b\sqrt{D} > 0, a - b\sqrt{D} > 0\}.$$

Theorem (Blomer-K 2015, K 2016)

For every M there are infinitely many $\mathbb{Q}(\sqrt{D})$ with no universal M-ary forms.

Tool 1: Indecomposable elements

$\alpha \in \mathcal{O}_K^+$ is *indecomposable* if $\alpha \neq \beta + \gamma$ for $\beta, \gamma \in \mathcal{O}_K^+$
Seems to be the key notion for studying universal forms!

Why useful?
$Q(X_1, \dots, X_r) = a_1 X_1^2 + a_2 X_2^2 + \cdots + a_r X_r^2$ universal diagonal form
$\alpha = a_1 x_1^2 + a_2 x_2^2 + \cdots + a_r x_r^2$ indecomposable
Thus $\alpha = a_i x_i^2$ is essentially one of the coefficients!

Used and studied in most of my papers on universal forms.

Also independently interesting:

Theorem (Hejda-K 2020)

The additive semigroup $\mathcal{O}_K^+(+)$ uniquely determines the real quadratic field $K = \mathbb{Q}(\sqrt{D})$.

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Tool 2: Continued fractions

Periodic continued fraction

$$\sqrt{D} = [u_0, \overline{u_1, u_2, \ldots, u_{s-1}, 2u_0}] = u_0 + \cfrac{1}{u_1 + \frac{1}{u_2 + \cdots}}.$$

Convergents $[u_0, u_1, u_2, \ldots, u_i]$ to the continued fraction give

- good approximations to $\sqrt{D}$ and
- indecomposables (elements hard to represent by a universal form).

Theorem (Blomer-K 2015, K 2016)

For every M there are infinitely many $\mathbb{Q}(\sqrt{D})$ with no universal M-ary forms.

If $s, u_1, u_2, \ldots$ large, then each universal form needs many variables.

Blomer-K 2018: Made more precise and related to class numbers.

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Higher degrees

Number field of degree n:
$K = \mathbb{Q}(\alpha)$ where $\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$
$m(K)$ = minimal number of variables of universal form

Complicated, because continued fractions don't work.
Partial results on large number of variables:

- Yatsyna (2019): $n = 3$
- K-Svoboda (2019): $n = 2^k$
- K (2021): n divisible by 2 or 3

Theorem (K-Tinková 2020)

Let $a \geq -1$ and consider simplest cubic field $K = \mathbb{Q}(\alpha)$ with $\alpha^3 - a\alpha^2 - (a+3)\alpha - 1 = 0$ (and $a^2 + 3a + 9$ squarefree). Then

$$\frac{\sqrt{a^2 + 3a + 8}}{3\sqrt{2}} < m(K) \leq 3(a^2 + 3a + 6).$$

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Outline

Lifting problem

$\mathbb{Z}$-form = positive definite form with $\mathbb{Z}$-coefficients

$$Q(X_1, \dots, X_r) = a_{11}X_1^2 + a_{12}X_1X_2 + a_{22}X_2^2 + \dots \text{ with } a_{ij} \in \mathbb{Z}$$

Question. (The lifting problem)
Can a $\mathbb{Z}$-form be universal over K?

Siegel: sum of squares NOT, unless $K = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$

Theorem (K-Yatsyna 2021)

K = totally real number field of degree n with a universal $\mathbb{Z}$-form.

- *If $n = 2$, then $K = \mathbb{Q}(\sqrt{5})$.*
- *If $n = 3, 4, 5, 7$ and K has principal codifferent ideal, then $K = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ (where $\zeta_7 = e^{2\pi i/7}$)*
- *$X^2 + Y^2 + Z^2 + W^2 + XY + XZ + XW$ is universal over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$*

Proof: geometry (quadratic lattices) and analysis (Dedekind zeta-function)

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Tool 3: Quadratic lattices

Lattice: eg. $\mathbb{Z}^2 \subset \mathbb{R}^2$
In general, have basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ and a lattice $L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n \subset \mathbb{R}^n$
Quadratic form: $Q(a_1, \ldots, a_n) = (\text{length}(a_1 v_1 + \cdots + a_n v_n))^2$
Q represents $k \in \mathbb{Z} \iff$ there is vector $v \in L$ of length $\sqrt{k}$

Minimal vectors are very important, ie.
$v \in L$ with minimal $Q(v) > 0$.
Lattice-based cryptography: it's hard to find the vector $v \in L$ that is closest to a given element of $\mathbb{R}^n$

Similarly for number field K, have $\mathcal{O}_K$-lattice $L = \mathcal{O}_K v_1 + \cdots + \mathcal{O}_K v_n \subset K^n$ and can use its geometry

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Tool 4: Zeta functions

Riemann zeta-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}} \text{ for real part of } s > 1$$

Dedekind zeta-function: generalization to number field K

$$\zeta_K(s) = \sum_I \frac{1}{N(I)^s} = \prod_P \frac{1}{1-N(P)^{-s}} \text{ for real part of } s > 1$$

Very important in analytic number theory (extended Riemann hypothesis!)

Siegel (1969): gave a formula for $\zeta_K(2)$ as a certain sum related to indecomposables (for $n = 2, 3, 4, 5, 7$):

$$\zeta_K(2) = (-1)^n \pi^{2n} 2^{n+2} b_n |\Delta_K|^{-3/2} \sum_{\alpha \in \mathcal{O}_K^{\vee,+}, Tr(\alpha)=1} \sigma((\alpha)(\mathcal{O}_K^{\vee})^{-1})$$

Cf. $\zeta(2) = \pi^2/6$.

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Conclusion

Combining these tools implies

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Also can solve "weak lifting problem":

Theorem (K-Yatsyna 2021+)

For each n, there are at most finitely many totally real number fields of degree n with a universal $\mathbb{Z}$-form.

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Future plans

Questions.

1. Kitaoka's Conjecture: There are only finitely many K with $m(K) = 3$
   partial progress, eg. Krásenský–Tinková–Zemková proved for $\mathbb{Q}(\sqrt{D}, \sqrt{E})$.
2. How does the minimal number of variables $m(K)$ depend on K?
   some understanding in deg 2 and special families in deg 3
3. Is there a variant of 290 Theorem?
   Yes, but what's the "290" constant for given K?
4. Can we relate $m(K), 290(K)$ to number field invariants (eg. regulator, class number) and determine their asymptotic behavior?
5. Are there infinitely many K with "small" class number?
   Very hard.

Want to solve these questions!

UFOCLAN

UFOCLAN: Universal quadratic FOrms and CLAss Numbers
Junior Star project from Czech Science Foundation GAČR for 2021–2025
4 postdocs + students

Main goals.

- describe indecomposables using generalized continued fractions
- characterize corresponding families of number fields
- apply the results to universal quadratic forms
- refine the theory of infrastructure in number fields to obtain new connection with class numbers

Combine algebraic, analytic, geometric, and computational methods

Thanks for your attention!

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