A NOTE ON BIORTHOGONAL SYSTEMS

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ABSTRACT. We consider the following problem (which is a generalisation of a folklore result Proposition 1 below): Given a continuous linear operator $T\colon X \to Y$, where Y is a Banach space with a (long) sub-symmetric basis, under which conditions can we find a continuous linear operator $S\colon X \to Y$ such that $S(B_X)$ contains the basis of Y. As a tool we also consider a non-separable version of [HJ, Theorem 3.56]: Given an infinite subset $A \subset X^*$, under which conditions can we find a biorthogonal system in $X \times A$ of cardinality $\operatorname{card} A$?

We are interested in a generalisation of the following two results into a non-separable and more general setting. The first one is a folklore result, see e.g. [HJ, Proposition 3.33]:

Proposition 1. *Let X be a Banach space, $Y = \ell_p$, $1 \le p < \infty$, or $Y = c_0$, and suppose there is a non-compact operator $T \in \mathcal{L}(X;Y)$. Then there are $S \in \mathcal{L}(X;Y)$ and a normalised basic sequence $\{x_n\} \subset X$ such that $S(x_n) = e_n$, $n \in \mathbb{N}$, where $\{e_n\}$ is the canonical basis of Y. If X does not contain ℓ_1, then $\{x_n\}$ may be chosen to be weakly null.*

It turns out that the following theorem by the authors which deals with finding biorthogonal systems in preduals is a good tool for this problem.

Theorem 2 ([HJ, Theorem 3.56]). *Let X be a Banach space and let $\{f_n\} \subset X^*$ be a bounded sequence. The following statements are equivalent:*

(i) $\{f_n\}$ is not a relatively compact set.
(ii) There are a subsequence $\{g_n\}$ of $\{f_n\}$ and an (infinite-dimensional) subspace $Y \subset X$ such that $\{g_n\restriction_Y\} \subset Y^*$ is a semi-normalised w^*-null sequence.
(iii) There is a semi-normalised basic sequence $\{x_n\} \subset X$ which is biorthogonal to a subsequence of $\{f_n\}$.

Moreover, we may assume in addition that $\{x_n\}$ is either weakly null or equivalent to the canonical basis of ℓ_1.

This theorem has useful applications, see e.g. [HJ] or [J].

To generalise these results into a non-separable setting we first need to define certain properties.

Let X be a normed linear space. For $A \subset X^*$ and $x \in X$ we denote $\operatorname{supp}_A x = \{f \in A;\ f(x) \neq 0\}$.

Definition 3. Let X be a normed linear space and $A \subset X^*$. We say that A has

- property $\mathcal{C}$ if $\operatorname{supp}_A x$ is countable for each $x \in X$;
- property $\mathcal{Z}$ if $f_n \overset{w^*}{\to} 0$ for every sequence $\{f_n\}$ of distinct elements of A;
- property $\mathcal{B}$ if for every $\varepsilon > 0$ there is $k(\varepsilon) \ge 0$ such that $\operatorname{card}\{f \in A;\ |f(x)| > \varepsilon\} \le k(\varepsilon)$ for any $x \in B_X$.

Let μ be an infinite cardinal. We say that X has

- property $\mathcal{C}_\mu$ if there is $A \subset S_{X^*}$ of cardinality μ with property $\mathcal{C}$;
- property $\mathcal{Z}_\mu$ if there is $A \subset S_{X^*}$ of cardinality μ with property $\mathcal{Z}$;
- property $\mathcal{B}_\mu$ if there is $A \subset S_{X^*}$ of cardinality μ with property $\mathcal{B}$.

For an application of property $\mathcal{Z}$ see e.g. [HŠZ], property $\mathcal{B}$ comes from [B], where it is used for a construction of smooth surjections, cf. [J]. First we gather some useful facts about these properties.

Fact 4. *Let X be a normed linear space and $A \subset X^*$. Then A has property $\mathcal{Z}$ if and only if for every $x \in X$ and every $\varepsilon > 0$ the set $\{f \in A;\ |f(x)| > \varepsilon\}$ is finite.*

Proof. $\Rightarrow$ Assume that there is $x \in X$ and $\varepsilon > 0$ such that $B = \{f \in A;\ |f(x)| > \varepsilon\}$ is infinite. Then there is a sequence $\{f_n\}$ of distinct elements of B and $|f_n(x)| > \varepsilon$ for each $n \in \mathbb{N}$, a contradiction with $\{f_n\}$ being w^*-null.

$\Leftarrow$ Let $\{f_n\}$ be a sequence of distinct elements of A, $x \in X$, and $\varepsilon > 0$. Then $B = \{n \in \mathbb{N};\ |f_n(x)| > \varepsilon\}$ is finite and $|f_n(x)| \le \varepsilon$ whenever $n > \max B$. □

In particular, note that $\mathcal{B} \Rightarrow \mathcal{Z} \Rightarrow \mathcal{C}$.

Further, note that every normed linear space of dimension at least 2 has trivially property $\mathcal{C}_\omega$. Moreover, by the Josefson-Nissenzweig theorem every infinite-dimensional normed linear space has even property $\mathcal{Z}_\omega$.

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Fact 5. *Let $\{(x_\gamma; f_\gamma)\}_{\gamma\in\Gamma}$ be a fundamental biorthogonal system in a normed linear space such that $\{f_\gamma\}_{\gamma\in\Gamma}$ is bounded. Then $\{f_\gamma;\ \gamma\in\Gamma\}$ has property $\mathcal{Z}$.*

Proof. Let $\{\gamma_n\}_{n=1}^\infty$ be a sequence of distinct elements of Γ. If $\gamma\in\Gamma$, then $f_{\gamma_n}(x_\gamma)\neq 0$ for at most one $n\in\mathbb{N}$ and hence $f_{\gamma_n}(x_\gamma)\to 0$. Consequently, $f_{\gamma_n}(y)\to 0$ for every $y\in Y=\operatorname{span}\{x_\gamma;\ \gamma\in\Gamma\}$. Let $C>0$ be such that $\|f_\gamma\|\le C$ for each $\gamma\in\Gamma$. Now given $x\in X$ and $\varepsilon>0$ we find $y\in Y$ such that $\|x-y\|<\frac{\varepsilon}{2C}$. Then there is $n_0\in\mathbb{N}$ such that $|f_{\gamma_n}(y)|<\frac{\varepsilon}{2}$ whenever $n\ge n_0$. So $|f_{\gamma_n}(x)|\le|f_{\gamma_n}(y)|+|f_{\gamma_n}(x-y)|<\varepsilon$ whenever $n\ge n_0$.

□

Fact 6. *Let Γ be any set, let M be a non-degenerate Orlicz function, and let $\{f_\gamma\}_{\gamma\in\Gamma}$ be the canonical coordinate functionals of the Orlicz space $h_M(\Gamma)$. Then $\{f_\gamma;\ \gamma\in\Gamma\}$ has property $\mathcal{B}$.*

Proof. We set $k(\varepsilon)=\frac{1}{M(\varepsilon)}$. Now if $x\in B_{h_M(\Gamma)}$, then $\sum_{\gamma\in\Gamma}M(|f_\gamma(x)|)\le 1$ and hence $\operatorname{card}\{\gamma\in\Gamma;\ |f_\gamma(x)|>\varepsilon\}<\frac{1}{M(\varepsilon)}=k(\varepsilon)$ for each $\varepsilon>0$.

□

Fact 7. *The properties $\mathcal{C}$, $\mathcal{Z}$, $\mathcal{B}$ are preserved by continuous linear mappings in the following sense: Let X, Y be normed linear spaces and $T\in\mathcal{L}(X;Y)$. If $A\subset Y^*$ has one of the properties $\mathcal{C}$, $\mathcal{Z}$, $\mathcal{B}$, then $T^*(A)$ has the same property.*

Moreover, if $T^*(A)\subset X^*\setminus\{0\}$ and A is infinite in case of property $\mathcal{Z}$ or $\mathcal{B}$, resp. uncountable in case of property $\mathcal{C}$, then $\operatorname{card}T^*(A)=\operatorname{card}A$.

Proof. The first statement follows from the fact that

$$\begin{aligned}\operatorname{card}\{g\in T^*(A);\ |g(x)|>\varepsilon\}&\le\operatorname{card}\{f\in A;\ |T^*(f)(x)|>\varepsilon\}=\operatorname{card}\{f\in A;\ |f(T(x))|>\varepsilon\}\\&=\operatorname{card}\left\{f\in A;\ \left|f\left(\tfrac{T(x)}{\|T\|}\right)\right|>\tfrac{\varepsilon}{\|T\|}\right\}\end{aligned}$$

for any $x\in X$ and $\varepsilon\ge 0$ (together with Fact 4).

To see the second statement, let $g\in T^*(A)$ and let $x\in X$ be such that $g(x)>0$. Then $\{f\in A;\ T^*(f)=g\}\subset\left\{f\in A;\ |T^*(f)(x)|>\frac{g(x)}{2}\right\}=\left\{f\in A;\ |f(T(x))|>\frac{g(x)}{2}\right\}$ and the last set is finite in case of property $\mathcal{Z}$ or $\mathcal{B}$, or countable in case of property $\mathcal{C}$. Since $A=\bigcup_{g\in T^*(A)}\{f\in A;\ T^*(f)=g\}$, it follows that $T^*(A)$ is infinite and $\operatorname{card}A\le\operatorname{card}T^*(A)$, and so $\operatorname{card}T^*(A)=\operatorname{card}A$.

□

Lemma 8. *Let X be a normed linear space and let $A\subset X^*$ be an infinite set with property $\mathcal{Z}$, resp. $\mathcal{B}$, and such that there is $\delta>0$ satisfying $\|f\|\ge\delta$ for each $f\in A$. Then $B=\left\{\frac{f}{\|f\|};\ f\in A\right\}\subset S_{X^*}$ has property $\mathcal{Z}$, resp. $\mathcal{B}$, and* $\operatorname{card}B=\operatorname{card}A$.

Proof. Consider the mapping $\Phi\colon A\to S_{X^*}$, $\Phi(f)=\frac{f}{\|f\|}$. Then $B=\Phi(A)$. Fix $x\in X$. Then $\{g\in B;\ |g(x)|>\varepsilon\}=\Phi(\{f\in A;\ |\Phi(f)(x)|>\varepsilon\})=\Phi(\{f\in A;\ |f(x)|>\varepsilon\|f\|\})\subset\Phi(\{f\in A;\ |f(x)|>\varepsilon\delta\})$ and so $\operatorname{card}\{g\in B;\ |g(x)|>\varepsilon\}\le\operatorname{card}\{f\in A;\ |f(x)|>\varepsilon\delta\}$. The last quantity is finite (Fact 4) and in case of property $\mathcal{B}$ and $x\in B_X$ not greater than $k(\varepsilon\delta)$. Consequently, B has property $\mathcal{Z}$, resp. $\mathcal{B}$.

Now let $g\in B$ and let $x\in X$ be such that $g(x)>0$. Then $\{f\in A;\ \Phi(f)=g\}\subset\left\{f\in A;\ |\Phi(f)(x)|>\frac{g(x)}{2}\right\}\subset\left\{f\in A;\ |f(x)|>\frac{g(x)}{2}\delta\right\}$ and the last set is finite. Since $A=\bigcup_{g\in B}\Phi^{-1}(g)$, it follows that B is infinite and $\operatorname{card}A\le\operatorname{card}B$, and so $\operatorname{card}B=\operatorname{card}A$.

□

The generalisation of Theorem 2 to large cardinalities is actually a purely combinatorial result:

Theorem 9. *Let X be a normed linear space and $A\subset X^*\setminus\{0\}$ a set with property $\mathcal{C}$ and $\operatorname{card}A>\omega_1$. Further, for each $f\in A$ let $x_f\in X$ be such that $f(x_f)=1$. Then there is $B\subset A$, $\operatorname{card}B=\operatorname{card}A$ such that $\{(x_f;f)\}_{f\in B}$ is a biorthogonal system.*

Proof. Define $F\colon A\to\mathcal{P}(A)$ by $F(f)=\operatorname{supp}_A x_f\setminus\{f\}$. Then by the assumption $\operatorname{card}F(f)<\omega_1$ for each $f\in A$. Therefore by Hajnal's theorem on free sets ([EHMR, Theorem 44.3]) there is $B\subset A$, $\operatorname{card}B=\operatorname{card}A$ that is free with respect to F, i.e. $F(f)\cap B=\emptyset$ for each $f\in B$. This means that $g\notin\operatorname{supp}_A x_f$, i.e. $g(x_f)=0$ for any $f,g\in B$, $f\neq g$.

□

For cardinality ω_1 we have at least the following well-known weaker statement.

Lemma 10. *Let X be a normed linear space and $A\subset X^*$ an uncountable set. Suppose that for each $f\in A$ there is a given $x_f\in X$ and that $\operatorname{supp}_A x_f$ is countable (this holds in particular if A has property $\mathcal{C}$). Then there is a long sequence $\{f_\alpha\}_{\alpha<\omega_1}\subset A$ such that $f_\beta(x_{f_\alpha})=0$ whenever $\alpha<\beta<\omega_1$.*

Proof. We will use transfinite induction. Let $\beta<\omega_1$. Then $\operatorname{card}\bigcup_{\alpha<\beta}\operatorname{supp}_A x_{f_\alpha}\le\max\{\operatorname{card}\beta,\omega\}<\omega_1$ and so there is $f_\beta\in A\setminus\bigcup_{\alpha<\beta}\operatorname{supp}_A x_{f_\alpha}$.

□

For cardinalities not larger than ω_1 it is already not a purely combinatorial problem and a bit of analysis gets involved. For the countable case see the proof of [HJ, Theorem 3.56]. For cardinality ω_1 we have the following proposition.

Proposition 11. *Let X be a Banach space and $A \subset X^*$. The following statements are equivalent:*

(i) There is a biorthogonal system $\{(x_\gamma; f_\gamma)\}_{\gamma\in\Gamma} \subset X \times A$ of cardinality ω_1.

(ii) There is a system $\{(z_\gamma; f_\gamma)\}_{\gamma\in\Gamma} \subset X \times A$ of cardinality ω_1 such that f_γs are distinct, $f_\alpha(z_\alpha) \neq 0$ and $\sum_{\gamma\in\Gamma}|f_\gamma(z_\alpha)| < +\infty$ for each $\alpha \in \Gamma$.

Proof. (i)⇒(ii) is trivial.

(ii)⇒(i) By scaling the z_γs we may assume that $f_\alpha(z_\alpha) = 1$ for each $\alpha \in \Gamma$. Further, by passing to a subset of Γ we may assume that $\{z_\gamma\}_{\gamma\in\Gamma}$ is bounded, say by $C > 0$. By Lemma 10 we may pass to a further subset of Γ and reindex by ordinals so that

$$f_\beta(z_\alpha) = 0 \quad \text{whenever } \alpha < \beta < \omega_1. \tag{1}$$

Pick any $\delta \in (0, 1)$. By the assumption for each $\alpha < \omega_1$ there is a finite set $F(\alpha) \subset \omega_1 \setminus \{\alpha\}$ such that $\sum_{\gamma\in\omega_1\setminus(F(\alpha)\cup\{\alpha\})}|f_\gamma(z_\alpha)| \leq \delta$. By Hajnal's theorem on free sets ([EHMR, Theorem 44.3]) there is $B \subset \omega_1$, card $B = \omega_1$ that is free with respect to F, i.e. $F(\alpha) \cap B = \emptyset$ for each $\alpha \in B$. By passing to this set B we may assume that

$$\sum_{\gamma<\alpha}|f_\gamma(z_\alpha)| \leq \delta \tag{2}$$

for each $\alpha < \omega_1$. (Note that we can retain the ordering during reindexing so that (1) still holds.)

Finally, we will construct a long sequence $\{x_\alpha\}_{\alpha<\omega_1} \subset X$ such that $\{(x_\alpha; f_\alpha)\}_{\alpha<\omega_1}$ is a biorthogonal system. Fix $\alpha < \omega_1$. By induction we construct a sequence $\{y_k\}_{k=1}^\infty \subset X$. Set $y_1 = z_\alpha$ and

$$y_{k+1} = y_k - \sum_{\gamma<\alpha} f_\gamma(y_k)z_\gamma$$

for $k \in \mathbb{N}$. The fact that y_k is well-defined for each $k \in \mathbb{N}$ will follow inductively from the following claim:

$$\sum_{\gamma<\alpha}|f_\gamma(y_k)| \leq \delta^k \tag{3}$$

for each $k \in \mathbb{N}$. For $k = 1$ it follows from (2). Now let $k \in \mathbb{N}$. If $\beta < \alpha$, then by the inductive hypothesis and (1) we obtain

$$f_\beta(y_{k+1}) = f_\beta(y_k) - \sum_{\gamma<\alpha} f_\gamma(y_k)f_\beta(z_\gamma) = -\sum_{\substack{\gamma<\alpha\\ \gamma\neq\beta}} f_\gamma(y_k)f_\beta(z_\gamma) = -\sum_{\beta<\gamma<\alpha} f_\gamma(y_k)f_\beta(z_\gamma).$$

Therefore

$$\begin{aligned}\sum_{\beta<\alpha}|f_\beta(y_{k+1})| &\leq \sum_{\beta<\alpha}\Bigg(\sum_{\beta<\gamma<\alpha}|f_\gamma(y_k)||f_\beta(z_\gamma)|\Bigg) = \sum_{\gamma<\alpha}\Bigg(\sum_{\beta<\gamma}|f_\gamma(y_k)||f_\beta(z_\gamma)|\Bigg) = \sum_{\gamma<\alpha}\Bigg(|f_\gamma(y_k)|\sum_{\beta<\gamma}|f_\beta(z_\gamma)|\Bigg)\\ &\leq \delta\sum_{\gamma<\alpha}|f_\gamma(y_k)| \leq \delta^{k+1},\end{aligned}$$

where we used the Fubini theorem for non-negative functions, (2), and the inductive hypothesis.

From (3) it follows that $\|y_{k+1} - y_k\| \leq \sum_{\gamma<\alpha}|f_\gamma(y_k)|\|z_\gamma\| \leq C\delta^k$. Consequently, $\|y_l - y_k\| \leq \sum_{j=k}^{l-1}\|y_{j+1} - y_j\| \leq C\sum_{j=k}^{l-1}\delta^j \leq \frac{C}{1-\delta}\delta^k$ for any $k, l \in \mathbb{N}$, $k < l$. Thus $\{y_k\}$ is Cauchy and hence convergent to some $x_\alpha \in X$. Using (1) and induction it is easily seen that $f_\alpha(y_k) = 1$ and $f_\beta(y_k) = 0$ whenever $\beta > \alpha$ and $k \in \mathbb{N}$. Thus $f_\alpha(x_\alpha) = 1$, while $f_\beta(x_\alpha) = 0$ for $\beta > \alpha$. Now if $\beta < \alpha$, then (3) implies that $|f_\beta(y_k)| \leq \delta^k$ for each $k \in \mathbb{N}$ and hence $f_\beta(x_\alpha) = \lim_{k\to\infty} f_\beta(y_k) = 0$. □

We note that the proof is based on a similar idea as in the countable case. The main difference is that in the countable case we easily get rid of the "head" of the vector and the analysis is used to get rid of the "tails", while in the the proof above it is the other way round.

Thanks to the results of Stevo Todorčević Proposition 11 can be applied in the case when we assume Martin's axiom MA_{ω_1}:

Theorem 12. (MA_{ω_1}) *Let X be a Banach space and $A \subset X^*$ a set with property $\mathcal{Z}$ and* card $A = \omega_1$. *Then there are semi-normalised systems $\{f_\alpha\}_{\alpha<\omega_1} \subset A$ and $\{x_\alpha\}_{\alpha<\omega_1} \subset X$ such that $\{(x_\alpha; f_\alpha)\}_{\alpha<\omega_1}$ is a biorthogonal system.*

Proof. Let $\{f_\alpha\}_{\alpha<\omega_1} \subset A \setminus \{0\}$ be such that all f_αs are distinct. For each $\alpha < \omega_1$ choose $z_\alpha \in X$ such that $f_\alpha(z_\alpha) = 1$. The axiom MA_{ω_1} allows us to pass to a subset of indices of cardinality ω_1 so that we may assume that $\sum_{\gamma<\omega_1}|f_\gamma(z_\alpha)| < +\infty$ for each $\alpha < \omega_1$. The proof can be found in [T, pp. 699–700] (we set $Z = \{z_\alpha;\ \alpha < \omega_1\}$), cf. also [HMVZ, pp. 153–154]. We conclude the proof by appealing to Proposition 11 and then passing to subsets of indices so that both $\{f_\alpha\}_{\alpha<\omega_1}$ and $\{x_\alpha\}_{\alpha<\omega_1}$ are semi-normalised. □

We note that Theorem 12 actually follows from the proof of [T, Theorem 1]. However, the argument using Proposition 11 as above is orders of magnitude simpler. Further, we remark that we do not know of any counterexample to a general version of Theorem 12 (without additional axioms), since the space $C(K)$, where K is a Kunen-type compact, does not have property $\mathcal{Z}_{\omega_1}$ ([HŠZ, Proposition 4]).

As we shall see, non-separable versions of Proposition 1 are tied with the behaviour of the dual operator. The next three propositions record some useful properties of the dual operator.

Proposition 13. *Let X, Y be normed linear spaces and $T \in \mathcal{L}(X;Y)$. Let $A \subset Y^*$ be a bounded infinite set of cardinality μ with property $\mathcal{C}$ if $\mu > \omega$ or $\mathcal{Z}$ if $\mu = \omega$. Then the following statements are equivalent:*

(i) There are $\delta > 0$ and $B \subset A$ with $\operatorname{card} B = \mu$ *such that $\|T^*(f)\| \geq \delta$ for every $f \in B$.*
(ii) Both $T(B_X)$ and $T^*(A)$ contain uniformly separated sets of cardinality μ.

Proof. (ii)⇒(i) If $T^*(B)$ is 2δ-separated, then the ball $U(0,\delta)$ contains at most one member of $T^*(B)$.

(i)⇒(ii) From Fact 7 we get that $\operatorname{card} T^*(B) = \operatorname{card} B = \mu$. Let $C > 0$ be such that $A \subset B(0,C)$. By Theorem 9 (for $\mu > \omega_1$), Lemma 10 (for $\mu = \omega_1$), or Theorem 2 (for $\mu = \omega$) there are $\{g_\alpha\}_{\alpha<\mu} \subset T^*(B)$ and $\{x_\alpha\}_{\alpha<\mu} \subset B_X(0,R)$ for some $R > 0$ such that $g_\alpha(x_\alpha) = 1$ and $g_\beta(x_\alpha) = 0$ whenever $\alpha < \beta < \mu$. For each $\alpha < \mu$ choose any $f_\alpha \in B$ such that $T^*(f_\alpha) = g_\alpha$. Then $\|g_\alpha - g_\beta\| \geq \frac{1}{\|x_\alpha\|}(g_\alpha - g_\beta)(x_\alpha) = \frac{1}{\|x_\alpha\|} \geq \frac{1}{R}$ and $\|T(x_\beta) - T(x_\alpha)\| \geq \frac{1}{\|f_\beta\|} f_\beta\bigl(T(x_\beta) - T(x_\alpha)\bigr) \geq \frac{1}{C}\bigl(T^*(f_\beta)(x_\beta) - T^*(f_\beta)(x_\alpha)\bigr) = \frac{1}{C}\bigl(g_\beta(x_\beta) - g_\beta(x_\alpha)\bigr) = \frac{1}{C}$ whenever $\alpha < \beta < \mu$.

□

Proposition 14. *Let X be a normed linear space, μ an infinite cardinal, and $T \in \mathcal{L}(X; c_0(\mu))$. Denote by $\{(e_\alpha; f_\alpha)\}_{\alpha<\mu}$ the canonical basis of $c_0(\mu)$. Then the following statements are equivalent:*

(i) $T^*(\{f_\alpha;\ \alpha < \mu\})$ contains a uniformly separated set of cardinality μ.
(ii) $T(B_X)$ contains a uniformly separated set of cardinality μ.

Proof. (i)⇒(ii) follows from Proposition 13 and Fact 5.

(ii)⇒(i) Let $A \subset B_X$ be such that $\operatorname{card} A = \mu$ and $\|T(x) - T(y)\| \geq \delta$ for some $\delta > 0$ and every $x, y \in A$, $x \neq y$. We will construct a sequence $\{\gamma_\alpha\}_{\alpha<\mu}$ of distinct ordinals such that $\|T^*(f_{\gamma_\alpha})\| \geq \frac{\delta}{2}$ by (transfinite) induction. The proof will then be finished by using Proposition 13 (and Fact 5). So let $\alpha < \mu$. Denote $\Gamma_\alpha = \{\gamma_\beta;\ \beta < \alpha\}$, $Z_\alpha = \overline{\operatorname{span}}\{e_\gamma;\ \gamma \in \Gamma_\alpha\}$, and $P_{\Gamma_\alpha}\colon c_0(\mu) \to Z_\alpha$ the canonical projection. Then $\operatorname{card}\Gamma_\alpha \leq \operatorname{card}\alpha$. Note that $P_{\Gamma_\alpha}(T(x)) \in B_{Z_\alpha}(0,\|T\|)$ for every $x \in A$. There are $x, y \in A$, $x \neq y$ such that $\|P_{\Gamma_\alpha}(T(x)) - P_{\Gamma_\alpha}(T(y))\| < \delta$. Indeed, if $\alpha < \omega$, then it follows from the compactness of $B_{Z_\alpha}(0,\|T\|)$, while for $\alpha \geq \omega$ it follows from the fact that $\operatorname{dens} B_{Z_\alpha}(0,\|T\|) \leq \operatorname{card}\alpha < \mu$. Since $\|T(x) - T(y)\| \geq \delta$, there is $\gamma_\alpha \in \mu \setminus \Gamma_\alpha$ such that $|f_{\gamma_\alpha}(T(x) - T(y))| \geq \delta$. Then $\|T^*(f_{\gamma_\alpha})\| \geq \frac{1}{2}|T^*(f_{\gamma_\alpha})(x-y)| = \frac{1}{2}|f_{\gamma_\alpha}(T(x-y))| \geq \frac{\delta}{2}$.

□

Proposition 15. *Let X be a normed linear space, $1 \leq p < \infty$, and μ an infinite cardinal. Denote by $\{(e_\alpha; f_\alpha)\}_{\alpha<\mu}$ the canonical basis of $\ell_p(\mu)$. Then the following statements are equivalent:*

(i) There is $T \in \mathcal{L}(X;\ell_p(\mu))$ such that $\{T^*(f_\alpha);\ \alpha < \mu\}$ is uniformly separated.
(ii) There is $T \in \mathcal{L}(X;\ell_p(\mu))$ such that $T(B_X)$ contains a uniformly separated set of cardinality μ.

Proof. (i)⇒(ii) follows from Proposition 13 and Fact 5.

(ii)⇒(i) Let $A \subset B_X$ be such that $\operatorname{card} A = \mu$ and $\|T(x) - T(y)\| \geq \delta$ for some $\delta > 0$ and every $x, y \in A$, $x \neq y$. Put $\varepsilon = \frac{\delta}{2}(1 - \frac{1}{2^p})^{\frac{1}{p}} > 0$. For any $\Lambda \subset \mu$ denote by $P_\Lambda\colon \ell_p(\mu) \to \overline{\operatorname{span}}\{e_\gamma;\ \gamma \in \Lambda\}$ the canonical restriction projection. We will construct (long) sequences $\{A_\alpha\}_{\alpha<\mu} \subset \mu$ of finite disjoint subsets of μ and $\{g_\alpha\}_{\alpha<\mu} \subset X^*$ by (transfinite) induction. Let $\alpha < \mu$. Denote $\Gamma_\alpha = \bigcup_{\beta<\alpha} A_\beta$ and $Z_\alpha = \overline{\operatorname{span}}\{e_\gamma;\ \gamma \in \Gamma_\alpha\}$. Then Γ_α is finite if $\alpha < \omega$ and $\operatorname{card}\Gamma_\alpha \leq \operatorname{card}\alpha$ otherwise. Note that $P_{\Gamma_\alpha}(T(x)) \in B_{Z_\alpha}(0,\|T\|)$ for every $x \in A$. There are $x, y \in A$, $x \neq y$ such that $\|P_{\Gamma_\alpha}(T(x)) - P_{\Gamma_\alpha}(T(y))\| < \frac{\delta}{2}$. Indeed, if $\alpha < \omega$, then it follows from the compactness of $B_{Z_\alpha}(0,\|T\|)$, while for $\alpha \geq \omega$ it follows from the fact that $\operatorname{dens} B_{Z_\alpha}(0,\|T\|) \leq \operatorname{card}\alpha < \mu$. Denote $z = x - y \in 2B_X$. Then $\delta^p \leq \|T(x) - T(y)\|^p = \|T(z)\|^p = \|P_{\Gamma_\alpha}(T(z))\|^p + \|P_{\mu\setminus\Gamma_\alpha}(T(z))\|^p < \frac{\delta^p}{2^p} + \|P_{\mu\setminus\Gamma_\alpha}(T(z))\|^p$ and hence $\|P_{\mu\setminus\Gamma_\alpha}(T(z))\| > \delta(1 - \frac{1}{2^p})^{\frac{1}{p}} = 2\varepsilon$. Thus there is $A_\alpha \subset \mu \setminus \Gamma_\alpha$ finite such that $\|P_{A_\alpha}(T(z))\| \geq 2\varepsilon$. Further, let $h_\alpha \in S_{\ell_p(A_\alpha)^*}$ be such that $h_\alpha\bigl(P_{A_\alpha}(T(z))\bigr) = \|P_{A_\alpha}(T(z))\| \geq 2\varepsilon$ and set $g_\alpha = h_\alpha \circ P_{A_\alpha} \circ T$. Then $g_\alpha \in X^*$, $\|g_\alpha\| \leq \|T\|$, and $\|g_\alpha\| \geq \frac{1}{2}g_\alpha(z) = \frac{1}{2}h_\alpha\bigl(P_{A_\alpha}(T(z))\bigr) \geq \varepsilon$. This finishes the induction step.

Now define $S\colon X \to \ell_\infty(\mu)$ by $S(x) = (g_\alpha(x))_{\alpha<\mu}$. Then S is clearly a linear mapping. Further, $\sum_{\alpha<\mu}|g_\alpha(x)|^p = \sum_{\alpha<\mu}\bigl|h_\alpha\bigl(P_{A_\alpha}(T(x))\bigr)\bigr|^p \leq \sum_{\alpha<\mu}\|P_{A_\alpha}(T(x))\|^p = \sum_{\alpha<\mu}\sum_{\gamma\in A_\alpha}|f_\gamma(T(x))|^p \leq \|T(x)\|^p$ for any $x \in X$. Consequently, S maps into $\ell_p(\mu)$ and $S \in (X;\ell_p(\mu))$. Moreover, $S^*(f_\alpha) = g_\alpha$, and so $\|S^*(f_\alpha)\| = \|g_\alpha\| \geq \varepsilon$ for each $\alpha < \mu$. We finish by applying Proposition 13 (passing to another restriction).

□

Finally, the next four theorems offer various variants of the generalisation of Proposition 1.

Theorem 16. *Let X be a Banach space and μ an infinite cardinal. Let Y be a Banach space with a (long) sub-symmetric Schauder basis $\{(e_\alpha; f_\alpha)\}_{\alpha<\mu}$. Consider the following statements:*

(i) There are $T \in \mathcal{L}(X;Y)$ and $\delta > 0$ such that $\|T^*(f_\alpha)\| \geq \delta$ for every $\alpha < \mu$.
(ii) There is $T \in \mathcal{L}(X;Y)$ such that $T(S_X) \supset \{e_\alpha\}_{\alpha<\mu}$.

Then (ii)⇒(i). If $\mu \neq \omega_1$ or if we assume MA_{ω_1}, then the statements are equivalent.

Proof. (ii)⇒(i) For each $\alpha < \mu$ let $x_\alpha \in B_X$ be such that $T(x_\alpha) = e_\alpha$. Then $\|T^*(f_\alpha)\| \geq T^*(f_\alpha)(x_\alpha) = f_\alpha(T(x_\alpha)) = f_\alpha(e_\alpha) = 1$.

Now assume that either $\mu \neq \omega_1$, or MA_{ω_1} holds and let us prove (i)⇒(ii). By Facts 5 and 7 the set $\{T^*(f_\alpha);\ \alpha < \mu\}$ has property $\mathcal{Z}$ and is of cardinality μ (note that $\{f_\alpha\}_{\alpha<\mu}$ is automatically semi-normalised, as the basis is sub-symmetric). Using Theorem 9 (for $\mu > \omega_1$), Theorem 12 (for $\mu = \omega_1$), or Theorem 2 (for $\mu = \omega$) we obtain $\Gamma \subset \mu$ with $\operatorname{card}\Gamma = \mu$ and a bounded $\{x_\gamma\}_{\gamma\in\Gamma} \subset X$ such that $\{(x_\gamma; T^*(f_\gamma))\}_{\gamma\in\Gamma}$ is a biorthogonal system. Denote $Z = \overline{\operatorname{span}}\{e_\gamma;\ \gamma \in \Gamma\}$. Define $S\colon X \to Z$ by $S(x) = \sum_{\gamma\in\Gamma}\|x_\gamma\| f_\gamma(T(x))e_\gamma$. The unconditionality of $\{e_\alpha\}_{\alpha<\mu}$ implies that S is a well-defined continuous linear mapping. Also, $S\bigl(\frac{x_\gamma}{\|x_\gamma\|}\bigr) = e_\gamma$ for each $\gamma \in \Gamma$. Finally, since $\{e_\alpha\}_{\alpha<\mu}$ is sub-symmetric, there is an isomorphism $I\colon Z \to Y$ that maps $\{e_\gamma\}_{\gamma\in\Gamma}$ onto $\{e_\alpha\}_{\alpha<\mu}$. The operator we seek in our proof is then $I \circ S$.

□

Theorem 17. *Let X be a normed linear space and μ an uncountable cardinal. Denote by $\{f_\alpha\}_{\alpha<\mu}$ the canonical coordinate functionals on $\ell_\infty^c(\mu)$ and by $\{e_\alpha\}_{\alpha<\mu}$ the canonical coordinate vectors in $\ell_\infty^c(\mu)$, i.e. $e_\alpha = \chi_{\{\alpha\}}$. Consider the following statements:*

(i) X has property $\mathcal{C}_\mu$.
(ii) There is $T \in \mathcal{L}(X; \ell_\infty^c(\mu))$ such that $T(B_X)$ contains a uniformly separated set of cardinality μ and $\{T^*(f_\alpha)\}_{\alpha<\mu} \subset S_{X^*}$ is uniformly separated.
(iii) There is $T \in \mathcal{L}(X; \ell_\infty^c(\mu))$ such that $\operatorname{card}\{\alpha;\ T^*(f_\alpha) \neq 0\} = \mu$.
(iv) There is $T \in \mathcal{L}(X; \ell_\infty^c(\mu))$ such that $T(X) \supset \{e_\alpha\}_{\alpha<\mu}$.
(v) There is $T \in \mathcal{L}(X; \ell_\infty^c(\mu))$ such that $T(S_X) \supset \{e_\alpha\}_{\alpha<\mu}$.

Then (i)⇔(ii)⇔(iii)⇐(iv)⇔(v). If $\mu > \omega_1$, then all the statements are equivalent.

Proof. (i)⇒(ii) Let $A \subset S_{X^*}$ be a set of cardinality μ with property $\mathcal{C}$. For each $f \in A$ choose any $x_f \in 2B_X$ such that $f(x_f) = 1$. By Theorem 9, resp. Lemma 10 there is $\{g_\alpha\}_{\alpha<\mu} \subset A$ such that $g_\beta(x_{g_\alpha}) = 0$ whenever $\alpha < \beta < \mu$. Define $T\colon X \to \ell_\infty(\mu)$ by $T(x) = (g_\alpha(x))_{\alpha<\mu}$. Property $\mathcal{C}$ implies that T actually maps into $\ell_\infty^c(\mu)$ and it is clearly a bounded linear operator. Note that $T^*(f_\alpha) = g_\alpha \in S_{X^*}$ for every $\alpha < \mu$. Further, $\|T(\frac12 x_{g_\alpha}) - T(\frac12 x_{g_\beta})\| \geq \bigl|f_\beta\bigl(T(\frac12 x_{g_\alpha}) - T(\frac12 x_{g_\beta})\bigr)\bigr| = \bigl|g_\beta(\frac12 x_{g_\alpha}) - g_\beta(\frac12 x_{g_\beta})\bigr| = \frac12$ and $\|T^*(f_\alpha) - T^*(f_\beta)\| = \|g_\alpha - g_\beta\| \geq (g_\alpha - g_\beta)(\frac12 x_{g_\alpha}) = \frac12$ whenever $\alpha < \beta < \mu$.

(ii)⇒(iii) is trivial.

(iii)⇒(i) Let $\Gamma = \{\alpha;\ T^*(f_\alpha) \neq 0\}$ and consider the mapping $F\colon \Gamma \to S_{X^*}$, $F(\alpha) = \frac{T^*(f_\alpha)}{\|T^*(f_\alpha)\|}$. Set $A = F(\Gamma)$. Given $g \in A$ and $x \in X$, note that $g(x) \neq 0$ if and only if $f_\alpha(T(x)) = T^*(f_\alpha)(x) \neq 0$ for every $\alpha \in F^{-1}(g)$. It follows that $\operatorname{supp}_A x \subset F(\operatorname{supp} T(x))$ and so $\operatorname{supp}_A x$ is countable for each $x \in X$. Hence A has property $\mathcal{C}$. It also follows that $F^{-1}(g)$ is countable for each $g \in A$. Consequently, $\operatorname{card}\Gamma = \operatorname{card}\bigcup_{g\in A} F^{-1}(g) \leq \max\{\operatorname{card}A, \omega\}$, and so $\operatorname{card}A = \operatorname{card}\Gamma = \mu$.

(iv)⇒(iii) For each $\alpha < \mu$ let $x_\alpha \in X$ be such that $T(x_\alpha) = e_\alpha$. Then $T^*(f_\alpha)(x_\alpha) = f_\alpha(T(x_\alpha)) = f_\alpha(e_\alpha) = 1$.

(v)⇒(iv) is trivial.

(iv)⇒(v) Assume that $\mu = \omega_1$. For each $\alpha < \mu$ let $x_\alpha \in X$ be such that $T(x_\alpha) = e_\alpha$. Consider $A_n = \{\alpha < \mu;\ \|x_\alpha\| \leq n\}$. Then $\mu = \bigcup_{n=1}^\infty A_n$ and since $\operatorname{cf}\mu > \omega$, there is $n \in \mathbb{N}$ such that $\operatorname{card}A_n = \mu$. Define $S\colon X \to \ell_\infty^c(A_n)$ by $S(x) = \bigl(\|x_\alpha\| f_\alpha(T(x))\bigr)_{\alpha\in A_n}$. Then $S \in \mathcal{L}(X; \ell_\infty^c(A_n))$, $\|S\| \leq n\|T\|$, and $S\bigl(\frac{x_\alpha}{\|x_\alpha\|}\bigr) = e_\alpha$ for each $\alpha \in A_n$.

Now assume that $\mu > \omega_1$ and let us prove (i)⇒(v). Let $A \subset S_{X^*}$ be a set of cardinality μ with property $\mathcal{C}$. For each $g \in A$ find $x_g \in 2B_X$ such that $g(x_g) = 1$. By Theorem 9 there is $B \subset A$ with $\operatorname{card}B = \mu$ such that $\{(x_g; g)\}_{g\in B}$ is a biorthogonal system. Define $T\colon X \to \ell_\infty(B)$ by $T(x) = \bigl(\|x_g\| g(x)\bigr)_{g\in B}$. Then clearly T is a bounded linear operator with $\|T\| \leq 2$ and by property $\mathcal{C}$ it maps into $\ell_\infty^c(B)$. Finally, $T\bigl(\frac{x_g}{\|x_g\|}\bigr) = e_g$ for each $g \in B$.

□

Corollary 18. *Let X be a Banach space and μ an infinite cardinal. Denote by $\{(e_\alpha; f_\alpha)\}_{\alpha<\mu}$ the canonical basis of $c_0(\mu)$. Consider the following statements:*

(i) X has property $\mathcal{Z}_\mu$.
(ii) There are $T \in \mathcal{L}(X; c_0(\mu))$ and $\delta > 0$ such that $\|T^*(f_\alpha)\| \geq \delta$ for every $\alpha < \mu$.
(iii) There is $T \in \mathcal{L}(X; c_0(\mu))$ such that $T(B_X)$ contains a uniformly separated set of cardinality μ.
(iv) There is $T \in \mathcal{L}(X; c_0(\mu))$ such that $T(S_X) \supset \{e_\alpha\}_{\alpha<\mu}$.

Then (i)⇔(ii)⇔(iii)⇐(iv). If $\mu \neq \omega_1$ or if we assume MA_{ω_1}, then all the statements are equivalent.

Proof. (i)⇒(ii) Let $A \subset S_{X^*}$ be a set of cardinality μ with property $\mathcal{Z}$. Let $\{g_\alpha\}_{\alpha<\mu} \subset A$ be a long sequence of distinct elements of A. Define $T\colon X \to \ell_\infty(\mu)$ by $T(x) = (g_\alpha(x))_{\alpha<\mu}$. Fact 4 implies that T actually maps into $c_0(\mu)$ and it is clearly a bounded linear operator. Obviously $T^*(f_\alpha) = g_\alpha \in S_{X^*}$ for every $\alpha < \mu$.

(ii)⇒(i) follows from Fact 5, Fact 7, and Lemma 8.

(ii)⇒(iii) follows from Fact 5 and Proposition 13.

(iii)⇒(ii) From Proposition 14 it follows that there are $\delta > 0$ and $\Gamma \subset \mu$ with $\operatorname{card}\Gamma = \mu$ such that $\|T^*(f_\gamma)\| \geq \delta$ for each $\gamma \in \Gamma$. Define a bounded linear operator $S\colon X \to c_0(\Gamma)$ by $S(x) = \bigl(f_\gamma(T(x))\bigr)_{\gamma\in\Gamma}$. Then clearly $S^*(f_\gamma) = T^*(f_\gamma)$ for each $\gamma \in \Gamma$.

The rest follows from Theorem 16.

□

Corollary 19. *Let X be a Banach space and μ an infinite cardinal. Denote by $\{(e_\alpha; f_\alpha)\}_{\alpha<\mu}$ the canonical basis of the Orlicz space $h_M(\mu)$. Consider the following statements:*

(i) X has property $\mathcal{B}_\mu$.
(ii) There are a non-degenerate Orlicz function M, $T \in \mathcal{L}(X; h_M(\mu))$, and $\delta > 0$ such that $\|T^*(f_\alpha)\| \geq \delta$ for every $\alpha < \mu$.
(iii) There are a non-degenerate Orlicz function M and $T \in \mathcal{L}(X; h_M(\mu))$ such that $T(S_X) \supset \{e_\alpha\}_{\alpha<\mu}$.

Then (i)⇔(ii)⇐(iii). If $\mu \neq \omega_1$ or if we assume MA_{ω_1}, then all the statements are equivalent.

Moreover, the Orlicz function M in (ii) and (iii) can be the same.

Proof. (i)⇒(ii) Let $A \subset S_{X^*}$ be a set of cardinality μ with property $\mathcal{B}$. Let $\{g_\alpha\}_{\alpha<\mu} \subset A$ be a long sequence of distinct elements of A. We may assume without loss of generality that the function $\varepsilon \mapsto k(\varepsilon)$ is positive and non-increasing. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ be any sequence decreasing to 0. Let $\varphi\colon [0,\varepsilon_1] \to \mathbb{R}$ be a function affine on each $[\varepsilon_{n+1}, \varepsilon_n]$ and satisfying $\varphi(\varepsilon_n) = \frac{1}{n^2} \cdot 1/k\left(\frac{\varepsilon_{n+1}}{n+1}\right)$, $\varphi(0) = 0$. Let M be the convex envelope of φ. It is easily seen that M can be extended to a non-degenerate Orlicz function. We define $T\colon X \to \ell_\infty(\mu)$ by $T(x) = (g_\alpha(x))_{\alpha<\mu}$. Then T is clearly a linear operator. Further, if $x \in B(0,\rho)$, then $\operatorname{card}\{\alpha < \mu;\ |g_\alpha(x)| > \varepsilon\} = \operatorname{card}\{\alpha < \mu;\ |g_\alpha(\frac{x}{\rho})| > \frac{\varepsilon}{\rho}\} \leq k(\frac{\varepsilon}{\rho})$, and so if $\rho \geq 1$, then (putting $\varepsilon_0 = \rho$)

$$\sum_{\alpha<\mu} M(|g_\alpha(x)|) = \sum_{n=1}^\infty \sum_{\{\alpha;\ \varepsilon_n<|g_\alpha(x)|\leq\varepsilon_{n-1}\}} M(|g_\alpha(x)|) \leq \sum_{n=1}^\infty M(\varepsilon_{n-1})k\left(\frac{\varepsilon_n}{\rho}\right) \leq \sum_{n\leq\rho} M(\varepsilon_{n-1})k\left(\frac{\varepsilon_n}{\rho}\right) + \sum_{n>\rho} M(\varepsilon_{n-1})k\left(\frac{\varepsilon_n}{n}\right)$$
$$\leq \sum_{n\leq\rho} M(\varepsilon_{n-1})k\left(\frac{\varepsilon_n}{\rho}\right) + \sum_{n>\rho} \frac{1}{(n-1)^2} < +\infty.$$

It follows that T actually maps into $h_M(\mu)$ and that $T \in \mathcal{L}(X; h_M(\mu))$. Clearly $T^*(f_\alpha) = g_\alpha \in S_{X^*}$ for every $\alpha < \mu$.

(ii)⇒(i) follows from Fact 6, Fact 7, and Lemma 8.

The rest follows from Theorem 16.

□

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