A FACTORISATION APPROACH TO BATES'S THEOREM

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ABSTRACT. We give a proof of theorem of S. M. Bates on smooth surjections between separable Banach spaces using a factorisation through an Orlicz sequence space. We believe that this approach is less technical and more transparent than the original proof.

It was shown by Sean Michael Bates [B] that every separable Banach space Y is a range of a C^1-smooth surjection $f\colon X \to Y$ from any infinite dimensional Banach space X. Moreover, if the space X has property $\mathcal{B}$ (defined below), then the surjection can be C^∞-smooth with all derivatives bounded on bounded sets. Independently, Petr Hájek [H2] showed that in certain cases the surjection can be even a polynomial. More precisely, he used the well-known fact that any separable Banach space is a quotient of ℓ_1 and then a factorisation

$$X \xrightarrow{T} \ell_p \xrightarrow{P} \ell_1 \xrightarrow{S} Y,$$

where S is a bounded linear surjection, P is a polynomial, and T is a non-compact bounded linear operator. One of the key observations is that there is a rather thin set $A \subset \ell_1$ such that $S(A)$ still contains a unit ball and A is in the image of $P \circ T$. The condition for the existence of a polynomial surjection is then the existence of an operator T above.

Inspired by Hájek's approach we show that there are even thinner sets in ℓ_1 still leading to a surjection (Corollary 4), which combined with the observation that property $\mathcal{B}$ gives a non-compact linear operator into an Orlicz space h_M (Proposition 2) leads to the factorisation

$$X \xrightarrow{T} h_M \xrightarrow{\Phi} \ell_1 \xrightarrow{S} Y,$$

where Φ is a rather simple C^∞-smooth mapping (Lemma 6). Thus we obtain a different, and perhaps easier, proof of Bates's theorem (Corollary 7). We do not claim that the proof is extremely short, but we believe that it is rather straightforward, transparent, and mostly non-technical, except perhaps the "insertion" Lemma 5, which belongs to the analysis of functions of one variable. In fact, the proof could be considerably condensed by putting all the intermediate lemmata together and disposing of the notion and theory of Orlicz spaces altogether, but that would only obscure the main ideas that are ultimately combined, and which can be perhaps of independent interest.

First we fix some notation. Let X be a normed linear space. By $B(x,r)$, resp. $U(x,r)$ we denote the closed, resp. open ball in X centred at $x \in X$ with radius $r \geq 0$. It will be convenient to use $U(x,+\infty) = B(x,+\infty) = X$. If X has a Schauder basis $\{e_n\}$ and $x = \sum_{n=1}^\infty x_n e_n \in X$, then by $\operatorname{supp} x$ we denote the support of x, i.e. the set $\{n \in \mathbb{N};\ x_n \neq 0\}$. If $\{x_n\}$ is a sequence in X, then $x_n(j)$ denotes the jth coordinate of $x_n \in X$.

By $D^k f(x)$ we denote the kth Fréchet derivative of f at x and by $D^k f(x)[h_1,\dots,h_k]$ we denote its evaluation at the directions $h_1,\dots,h_k$. By $C^\infty(X;Y)$ we denote the space of C^∞-smooth mappings from X to Y and by $\mathcal{C}^\infty(X;Y)$ we denote the space of C^∞-smooth mappings from X to Y that have all derivatives bounded on bounded sets.

Next we recall some basic facts about Orlicz sequence spaces h_M. A function $M\colon \mathbb{R} \to [0,+\infty)$ is called an Orlicz function if it is even, convex, non-decreasing on $[0,+\infty)$, $M(0) = 0$, and if M is not constant. An Orlicz sequence space h_M is a linear subspace of ℓ_∞ (with real or complex scalars) consisting of $x \in \ell_\infty$ satisfying $\sum_{n=1}^\infty M(|x_n|/\rho) < +\infty$ for all $\rho > 0$, equipped with the norm given by the Minkowski functional of $\{x \in h_M;\ \sum_{n=1}^\infty M(|x_n|) \leq 1\}$. With this norm the space h_M is a Banach space and the canonical basis vectors $\{e_n\}_{n=1}^\infty$ form a symmetric Schauder basis of h_M. Further, $x \mapsto \sum_{n=1}^\infty M(|x_n|)$ is continuous on h_M and $\sum_{n=1}^\infty M(|x_n|) \leq \|x\|$ for $x \in B_{h_M}$, while $\sum_{n=1}^\infty M(|x_n|) \geq \|x\|$ for $x \in h_M$, $\|x\| > 1$. If M is such that $M(t) = 0$ for some $t > 0$, then it is called a degenerate Orlicz function. The associated space h_M is then isomorphic to c_0. In connection with this and Proposition 2 we remark that c_0 does not have property $\mathcal{B}$ ([B], cf. [HJ, Proposition 6.38]) and there is no C^2-smooth surjection from c_0 onto ℓ_2 ([H1], see also [HJ, Chapter 6]). For more on the Orlicz sequence spaces see e.g. [LT, Chapter 4], for their smoothness properties see [HJ, Section 5.9].

Now we turn to producing the building blocks of the aforementioned factorisation.

Definition 1. We say that a Banach space X has property $\mathcal{B}$ if X^* contains a normalised sequence $\{f_n\}_{n=1}^\infty$ such that for every $\varepsilon > 0$ there is $k(\varepsilon) \geq 0$ such that $\operatorname{card}\{n \in \mathbb{N};\ |f_n(x)| > \varepsilon\} \leq k(\varepsilon)$ for any $x \in B_X$.

Note that the sequence $\{f_n\}$ from the definition above is in particular w^*-null.

A key observation that allows us to do the factorisation is the following.

Proposition 2. *A Banach space X has property $\mathcal{B}$ if and only if there are a non-degenerate Orlicz function M and a bounded linear operator $T\colon X \to h_M$ such that $T(B_X)$ contains the canonical basis of h_M.*

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Proof. $\Leftarrow$ Denote by $\{(e_n; f_n)\}_{n=1}^{\infty}$ the canonical basis of h_M. Note that if $y \in B_{h_M}$, then $\sum_{n=1}^{\infty} M(|f_n(y)|) \leq 1$ and hence $\operatorname{card}\{n;\ |f_n(y)| > \delta\} < 1/M(\delta)$ for each $\delta > 0$. Let $\{x_n\} \subset B_X$ be such that $T(x_n) = e_n$. Notice that $\|T^*(f_n)\| \geq T^*(f_n)(x_n) = f_n(T(x_n)) = f_n(e_n) = 1$. Put $g_n = T^*(f_n)/\|T^*(f_n)\|$. Now pick any $x \in B_X$. Then $|g_n(x)| = \left|\frac{T^*(f_n)(x)}{\|T^*(f_n)\|}\right| = \left|\frac{f_n(T(x))}{\|T^*(f_n)\|}\right| \leq |f_n(T(x))|$ and hence

$$\operatorname{card}\{n \in \mathbb{N};\ |g_n(x)| > \varepsilon\} \leq \operatorname{card}\{n \in \mathbb{N};\ |f_n(T(x))| > \varepsilon\} = \operatorname{card}\Big\{n \in \mathbb{N};\ \Big|f_n\Big(\frac{T(x)}{\|T\|}\Big)\Big| > \frac{\varepsilon}{\|T\|}\Big\} < \frac{1}{M\big(\frac{\varepsilon}{\|T\|}\big)},$$

since $\frac{T(x)}{\|T\|} \in B_{h_M}$.

$\Rightarrow$ We may assume without loss of generality that the function $\varepsilon \mapsto k(\varepsilon)$ is positive and non-increasing. Further, according to [HJ, Theorem 3.56] by passing to a subsequence we may assume that there is a semi-normalised sequence $\{x_n\} \subset X$ such that $\{(x_n; f_n)\}_{n=1}^{\infty}$ is a biorthogonal system. Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0,1)$ be any sequence decreasing to 0. Let $g\colon [0, \varepsilon_1] \to \mathbb{R}$ be a function affine on each $[\varepsilon_{n+1}, \varepsilon_n]$ and satisfying $g(\varepsilon_n) = \frac{1}{n^2} \cdot 1/k\big(\frac{\varepsilon_{n+1}}{n+1}\big)$, $g(0) = 0$. Let M be the convex envelope of g. It is easily seen that M can be extended to a non-degenerate Orlicz function. We define $T\colon X \to \ell_\infty$ by $T(x) = (f_n(x))_{n\in\mathbb{N}}$. Then T is clearly a linear operator. Further, if $x \in B(0,\rho)$, then $\operatorname{card}\{n \in \mathbb{N};\ |f_n(x)| > \varepsilon\} = \operatorname{card}\{n \in \mathbb{N};\ |f_n(\frac{x}{\rho})| > \frac{\varepsilon}{\rho}\} \leq k(\frac{\varepsilon}{\rho})$, and so if $\rho \geq 1$, then (putting $\varepsilon_0 = \rho$)

$$\begin{aligned}\sum_{n=1}^{\infty} M(|f_n(x)|) &= \sum_{n=1}^{\infty} \sum_{\{i;\ \varepsilon_n < |f_i(x)| \leq \varepsilon_{n-1}\}} M(|f_i(x)|) \leq \sum_{n=1}^{\infty} M(\varepsilon_{n-1}) k\Big(\frac{\varepsilon_n}{\rho}\Big) \leq \sum_{n\leq\rho} M(\varepsilon_{n-1}) k\Big(\frac{\varepsilon_n}{\rho}\Big) + \sum_{n>\rho} M(\varepsilon_{n-1}) k\Big(\frac{\varepsilon_n}{n}\Big)\\ &\leq \sum_{n\leq\rho} M(\varepsilon_{n-1}) k\Big(\frac{\varepsilon_n}{\rho}\Big) + \sum_{n>\rho} \frac{1}{(n-1)^2} < +\infty.\end{aligned}$$

It follows that T actually maps into h_M and that it is bounded. It is clear that T maps the vectors $\{x_n\}$ onto the canonical basis of h_M. By scaling T if necessary we can achieve that the canonical basis is contained in $T(B_X)$. $\square$

The following lemma, which is also behind the Open mapping theorem, is a slight modification of [HJ, Fact 6.64]; here we give "the right" formulation, although for the factorisation we do not need its full strength.

Lemma 3. *Let Y be a normed linear space over $\mathbb{K}$, $\{y_\gamma\}_{\gamma\in\Gamma}$ a dense subset of B_Y, and let $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{K} \setminus \{0\}$. For each $y \in U\big(0, \sum_{n=1}^{\infty}|\lambda_n|\big)$ there is a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of distinct elements of Γ such that $y = \sum_{n=1}^{\infty} \lambda_n y_{\gamma_n}$. If $\Gamma = \mathbb{N}$, then $\{\gamma_n\}$ can be chosen to be increasing.*

Proof. First we choose $N \in \mathbb{N}$ such that $R = \sum_{n=1}^{N}|\lambda_n| \geq \|y\|$. Since $\frac{|\lambda_n|}{\lambda_n R} y \in B_Y$, for each $n = 1, \dots, N$ we can find $\gamma_n \in \Gamma \setminus \{\gamma_1, \dots, \gamma_{n-1}\}$ such that $\big\|\frac{|\lambda_n|}{\lambda_n R} y - y_{\gamma_n}\big\| < \frac{|\lambda_{N+1}|}{N|\lambda_n|}$. Hence

$$\Big\|y - \sum_{n=1}^{N} \lambda_n y_{\gamma_n}\Big\| = \Big\|\frac{\sum_{n=1}^{N}|\lambda_n|}{R} y - \sum_{n=1}^{N} \lambda_n y_{\gamma_n}\Big\| = \Big\|\sum_{n=1}^{N} \lambda_n \frac{|\lambda_n|}{\lambda_n R} y - \sum_{n=1}^{N} \lambda_n y_{\gamma_n}\Big\| \leq \sum_{n=1}^{N} |\lambda_n| \Big\|\frac{|\lambda_n|}{\lambda_n R} y - y_{\gamma_n}\Big\| < |\lambda_{N+1}|. \quad (1)$$

We proceed by induction: we find a sequence $\{\gamma_n\}_{n=N+1}^{\infty} \subset \Gamma$ such that $\big\|y - \sum_{k=1}^{n} \lambda_k y_{\gamma_k}\big\| < \min\big\{|\lambda_{n+1}|, \frac{1}{n}\big\}$ for every $n > N$. Let $n > N$. Then $z = \frac{1}{\lambda_n}\big(y - \sum_{k=1}^{n-1} \lambda_k y_{\gamma_k}\big) \in B_Y$ by the inductive assumption (or by (1) if $n = N+1$) and so there is $\gamma_n \in \Gamma \setminus \{\gamma_1, \dots, \gamma_{n-1}\}$ such that $\|z - y_{\gamma_n}\| < \min\big\{|\lambda_{n+1}|, \frac{1}{n}\big\}/|\lambda_n|$, which finishes the construction. In case that $\Gamma = \mathbb{N}$ we can of course always choose $\gamma_n > \gamma_{n-1}$. $\square$

Let X be a Banach space with a sub-symmetric Schauder basis. To each $x = (x_n) \in X$ we associate a subset $S_x \subset X$ called the spreading of x by $S_x = \big\{\sum_{k=1}^{\infty} x_k e_{n_k};\ \{n_k\}_{k=1}^{\infty} \subset \mathbb{N} \text{ increasing}\big\}$. The next corollary is an improvement of the well-known fact that any separable Banach space is a quotient of ℓ_1.

Corollary 4. *Let Y be a separable Banach space. Then there exists a bounded linear operator $T\colon \ell_1 \to Y$ such that $T(S_x) \supset U(0, \|x\|)$ whenever $x \in \ell_1$ is such that* $\operatorname{supp} x = \mathbb{N}$.

Proof. Let $\{y_n\}$ be a dense subset of B_Y. Define $T\colon \ell_1 \to Y$ by $T(x) = \sum_{n=1}^{\infty} x_n y_n$. The conclusion now follows from Lemma 3. $\square$

The following lemma is an "insertion" type result that will be needed for the construction of the smooth mapping from h_M to ℓ_1.

Lemma 5. *Let $v_k\colon [0,+\infty) \to [0,+\infty)$, $k \in \mathbb{N}_0$, be non-decreasing functions positive on $(0,+\infty)$. There are an even $\varphi \in C^\infty(\mathbb{R})$ positive on $\mathbb{R} \setminus \{0\}$ and constants $C_k > 0$ such that $\big|\varphi^{(k)}(t)\big| \leq C_k v_k(|t|)$ for every $t \in \mathbb{R}$ and $k \in \mathbb{N}_0$. If moreover $\liminf\limits_{t\to+\infty} \frac{v_0(t)}{t} > 0$, then φ can be chosen so that $\lim\limits_{t\to+\infty} \varphi(t) = +\infty$.*

Proof. Without loss of generality we may assume that $\lim_{t\to 0+} v_k(t) = 0$ for each $k \in \mathbb{N}_0$. Let $\psi_n \in C^\infty(\mathbb{R})$ be non-negative, such that $\operatorname{supp}\psi_n = \left[\frac{1}{n+2}, \frac{1}{n}\right]$, $n \in \mathbb{N}_0$, and let ψ_0 be constant on $[1, +\infty)$. For each $n \in \mathbb{N}_0$ we find $a_n > 0$ such that

$$a_n \max\left|\psi_n^{(k)}\right| \le \frac{1}{2} v_k\left(\frac{1}{n+2}\right) \quad \text{for each } k \in \{0, \dots, n\}. \tag{2}$$

Next we put $C_0 = 1$ and for each $k \in \mathbb{N}$ we find $C_k \ge 1$ such that

$$\max\left|a_n \psi_n^{(k)}\right| \le \frac{1}{2} C_k v_k\left(\frac{1}{n+2}\right) \quad \text{for each } n \in \{0, \dots, k-1\}. \tag{3}$$

It follows that $\left|a_n \psi_n^{(k)}(t)\right| \le \frac{1}{2} C_k v_k(t)$ for every $t \ge 0$ and $n, k \in \mathbb{N}_0$: indeed, for a fixed n and $t \le \frac{1}{n+2}$ all the derivatives of ψ_n are zero, while for $t \ge \frac{1}{n+2}$ we use (2) for $k \le n$ and (3) for $k > n$. Now it suffices to put $\varphi(t) = \sum_{n=0}^\infty a_n \psi_n(|t|)$ for $t \in \mathbb{R}$. Then $\varphi^{(k)}(t) = a_n \psi_n^{(k)}(t) + a_{n+1}\psi_{n+1}^{(k)}(t)$ for $t \in \left[\frac{1}{n+2}, \frac{1}{n+1}\right]$, and so $\left|\varphi^{(k)}(t)\right| \le C_k v_k(|t|)$ for $t \ne 0$. From this estimate we also conclude by induction that $\varphi^{(k)}(0) = 0$ for $k \in \mathbb{N}_0$ and that $\varphi \in C^\infty(\mathbb{R})$.

Finally, suppose that moreover $\liminf_{t\to+\infty} \frac{v_0(t)}{t} > 0$. Then we can take ψ_0 to be affine and increasing on $[1, +\infty)$. We will also replace (2) for $n = 0$ by $|a_0 \psi_0(t)| \le \frac{1}{2} v_0(t)$ for $t \ge 0$. The rest of the proof is the same. □

Next we construct the main part of the factorisation.

Lemma 6. *Let M be a non-degenerate Orlicz function. Then there is a mapping $\Phi \in \mathcal{C}^\infty(h_M; \ell_1)$, where the spaces are real, with the following properties:*

(i) $\operatorname{supp}\Phi(x) = \operatorname{supp} x$ *for any* $x \in h_M$,
(ii) $\Phi(S_x) = S_{\Phi(x)}$ *for any* $x \in h_M$,
(iii) $\lim_{n\to\infty} \Phi(x_n)(1) = +\infty$ *whenever* $\{x_n\} \subset h_M$ *satisfies* $\lim_{n\to\infty} x_n(1) = +\infty$.

Proof. Without loss of generality we may assume that $M(1) = 1$. By Lemma 5 there are an even function $\varphi \in C^\infty(\mathbb{R})$ positive on $\mathbb{R} \setminus \{0\}$ with $\lim_{t\to+\infty} \varphi(t) = +\infty$, and constants $C_k > 0$ such that $|\varphi^{(k)}(t)| \le C_k M(\frac{t}{k+1})$ for every $t \in \mathbb{R}$, $k \in \mathbb{N}_0$. We define $\Phi\colon h_M \to \ell_1$ by $\Phi(x) = (\varphi(x_n))$. Notice that $\sum_{n=1}^\infty |\varphi(x_n)| \le C_0 \sum_{n=1}^\infty M(x_n) < +\infty$ for any $x = (x_n) \in h_M$ and so indeed Φ maps into ℓ_1. Moreover, Φ is locally bounded, since the estimate on the right-hand side is continuous on h_M. The mapping Φ clearly has properties (i)–(iii).

To see the smoothness of Φ, consider $\varepsilon = (\varepsilon_n)_{n=1}^N \in \{-1, 1\}^N \subset \ell_1^*$. Let $\{f_n\}$ be the canonical coordinate functionals on h_M and notice that $\|f_n\| = 1$. Then $\varepsilon \circ \Phi = \sum_{n=1}^N \varepsilon_n \varphi \circ f_n$ is clearly C^∞-smooth and $D^k(\varepsilon \circ \Phi)(x)[h_1, \dots, h_k] = \sum_{n=1}^N \varepsilon_n \varphi^{(k)}(f_n(x)) \cdot f_n(h_1) \cdots f_n(h_k)$ ([HJ, Corollary 1.117]). Therefore, for a fixed $k \in \mathbb{N}$ and $x \in U(0, k)$ we have $\|D^k(\varepsilon \circ \Phi)(x)\| \le \sum_{n=1}^N |\varphi^{(k)}(f_n(x))| \le C_k \sum_{n=1}^N M\left(\frac{f_n(x)}{k+1}\right) \le C_k$. Consequently, $D^{k-1}(\varepsilon \circ \Phi)$ is C_k-Lipschitz on $U(0, k)$. Since all such ε form a norming set for ℓ_1, it follows that Φ is C^{k-1}-smooth with the $(k-1)$th derivative Lipschitz (and hence bounded) on $U(0, k)$, [HJ, Corollary 1.131]. As this holds for every $k \in \mathbb{N}$, we finally conclude that $\Phi \in \mathcal{C}^\infty(h_M; \ell_1)$. □

Finally, we can put all together to recover Bates's theorem.

Corollary 7 (S. M. Bates)**.** *If X is a real Banach space with property $\mathcal{B}$ then for any separable Banach space Y there is a surjection $f \in \mathcal{C}^\infty(X; Y)$.*

Proof. Let $T\colon X \to h_M$ be the linear operator from Proposition 2, let $\Phi\colon h_M \to \ell_1$ be the smooth mapping from Lemma 6, and let $V\colon \ell_1 \to Y$ be the linear operator from Corollary 4. We set $f = V \circ \Phi \circ T$. Then $f \in \mathcal{C}^\infty(X; Y)$ by [HJ, Proposition 1.128]. Now let $\{x_n\} \subset B_X$ be such that $T(x_n) = e_n$, where $\{e_n\}$ is the canonical basis of h_M. Put $v = \sum_{n=1}^\infty \frac{1}{2^n} e_n$ and $A = \left\{\sum_{k=1}^\infty \frac{1}{2^k} x_{n_k};\ \{n_k\}_{k=1}^\infty \subset \mathbb{N} \text{ increasing}\right\}$. Note that $S_{nv} = T(nA)$ for $n \in \mathbb{N}$. Further, denote $z_n = \Phi(nv)$. Then $\operatorname{supp} z_n = \operatorname{supp} nv = \mathbb{N}$, $S_{z_n} = \Phi(S_{nv}) = \Phi(T(nA))$, and $z_n(1) \to +\infty$ as $nv(1) \to +\infty$. Consequently,

$$Y = \bigcup_{n=1}^\infty U\big(0, \|z_n\|\big) \subset \bigcup_{n=1}^\infty V(S_{z_n}) = \bigcup_{n=1}^\infty V\big(\Phi\big(T(nA)\big)\big) = \bigcup_{n=1}^\infty f(nA).$$

□

We close the paper by a few final remarks. When we write down the composition of the constructed mappings, the final formula for the surjection is

$$f(x) = \sum_{n=1}^\infty \varphi(f_n(x)) y_n,$$

where $\{y_n\}$ is a dense subset of B_Y and $f_n \in X^*$ are suitably chosen functionals so that f is onto. It is easily checked that the derivative of f is given by $Df(x)[h] = \sum_{n=1}^\infty \varphi'(f_n(x)) f_n(h) y_n$. Therefore the derivative at most points (those with sufficiently many non-zero f_n-coordinates) has infinite-dimensional range. This is in sharp contrast with the original proof of Bates, where he explicitly stresses that his surjection is highly singular: its derivative is of rank at most 1 at every point.

The last remark concerns non-separable spaces. The original paper contains also a non-separable version of Corollary 7: Let μ be an infinite cardinal. If X is a real Banach space with property $\mathcal{B}_\mu$, then for any Banach space Y of density at most μ there is a surjection $f \in \mathcal{C}^\infty(X;Y)$. (We say that a Banach space X has property $\mathcal{B}_\mu$ if there is $A \subset S_{X^*}$ of cardinality μ such that for every $\varepsilon > 0$ there is $k(\varepsilon) \geq 0$ such that $\operatorname{card}\{f \in A;\ |f(x)| > \varepsilon\} \leq k(\varepsilon)$ for any $x \in B_X$. We note that the property $\mathcal{B}_\mu$ is called $(\mu)_\infty$ in the Bates's paper and that the separable version is perhaps better known in the formulation due to E. Odell and H. Rosenthal: the property $\mathcal{B} = \mathcal{B}_\omega$ is equivalent to the existence of a normalised weakly null hereditarily Banach-Saks sequence in X^*, see [B, Lemma 3.2].)

Our factorisation approach works almost verbatim also in this non-separable setting except for the crucial step in the proof of Proposition 2, where we use [HJ, Theorem 3.56] to obtain the canonical basis in the range of T. To replicate this step we would need a non-separable version of [HJ, Theorem 3.56]. This version is true for $\mu > \omega_1$, but unfortunately at present we can show it for $\mu = \omega_1$ only under additional set-theoretic axiom MA_{ω_1}. The details can be found in [HJ2].

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