A REMARK ON SMOOTH IMAGES OF BANACH SPACES

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ABSTRACT. Let X be a Banach space with a non-separable super-reflexive quotient. Then for any separable Banach space Y of dimension at least two there exists a C^∞-smooth surjective mapping $f: X \to Y$ such that the restriction of f onto any separable subspace of X fails to be surjective. This solves a problem posed by Aron, Jaramillo, and Ransford (Problem 186 in the book [GMZ]).

1. SMOOTH IMAGES

It was shown by Bates [B] that every separable Banach space Y is a range of a C^1-smooth surjection $f: X \to Y$ from any infinite dimensional separable Banach space X. Moreover, under rather general conditions f can be chosen to be C^∞-smooth. On the other hand, it was shown in [Há1] that if $X = c_0$ and $Y = \ell_2$, then f cannot be C^2-smooth. In the setting of the non-separable space $X = c_0(\omega_1)$ it turns out ([GHM]) that the existence of C^2-smooth surjections onto ℓ_2 depends on the additional axioms of set theory. In particular, under the continuum hypothesis such surjections easily exist, while under the Martin's axiom MA_{ω_1} (which implies the negation of the continuum hypothesis) there are no such C^2-smooth surjections.

In a recent paper of Aron, Jaramillo, and Ransford, [AJR], the following result was shown: Let Γ be a set of cardinality at least continuum $\mathfrak{c}$, and suppose there exists a bounded linear operator $L: X \to c_0(\Gamma)$, such that $L(X)$ contains the canonical basis of $c_0(\Gamma)$. Then for any separable Banach space Y of dimension at least two there exists a C^∞-smooth surjective mapping $f: X \to Y$ such that the restriction of f onto any separable subspace of X fails to be surjective.

Of course, the result applies in particular to spaces $X = c_0(\mathfrak{c})$ or $X = \ell_p(\mathfrak{c})$, $1 \le p < \infty$. As we have seen above, this result holds for $c_0(\omega_1)$ if we assume the continuum hypothesis, but it fails under MA_{ω_1}, as ℓ_2 cannot be a range of a C^∞-smooth surjection. It is therefore quite natural to ask what happens for $X = \ell_p(\omega_1)$ spaces in this respect. The authors in [AJR] speculate that the result perhaps again depends on additional axioms of set theory. The problem is posed also in the recent monograph of Guirao, Montesinos, and Zizler, [GMZ, Problem 186].

Corollary 5 in this note gives a solution to this problem. In particular, there is a C^∞-smooth surjection $f: \ell_p(\omega_1) \to Y$ onto any separable Banach space Y of dimension at least 2 such that the restriction of f onto any separable subspace of $\ell_p(\omega_1)$ fails to be surjective. Our result is in fact somewhat more general, and applies in particular to all non-separable super-reflexive Banach spaces.

Before formulating the main theorem we recall some basic definitions on trees. A tree is a partially ordered set $(T, \preceq)$ with the property that for every $t \in T$ the subset $\{s \in T;\ s \preceq t\}$ is well-ordered. For $t \in T$ we denote by t^+ the set of all immediate successors of t, i.e. $t^+ = \{u \in T;\ s \prec u$ if and only if $s \preceq t\}$. For $u \in T$ we write u^- for the unique $t \in T$ such that $u \in t^+$, if such t exists. If a tree has a least element, then we will call this tree rooted. We will assume that the least element of a rooted tree is designated by 0, unless stated otherwise. The height of an element $t \in T$ is a unique ordinal $\mathrm{ht}(t)$ with the same order type as $\{s \in T;\ s \prec t\}$. The height of the tree T is defined by $\sup\{\mathrm{ht}(t) + 1;\ t \in T\}$. A branch of T is a maximal linearly ordered subset and we denote by $\mathcal{B}(T)$ the set of all branches of T. For an ordinal α we denote by $T_\alpha = \{t \in T;\ \mathrm{ht}(t) = \alpha\}$ the αth level of the tree T. For a branch $b \in \mathcal{B}(T)$ we denote $b_\alpha = b \cap T_\alpha$. Let μ be a cardinal. We say that T is μ-branching if $\operatorname{card} T_0 \le \mu$ and $\operatorname{card} t^+ \le \mu$ for each $t \in T$.

Let μ be a cardinal. We say that a subset S of a topological space X is μ-Suslin in X if there is a μ-branching tree T of height ω and closed sets $F_t \subset X$, $t \in T$ such that $S = \bigcup_{b \in \mathcal{B}(T)} \bigcap_{n=1}^\infty F_{b_n}$. We remark that ω-Suslin sets are called simply Suslin in the classical terminology and that a classical result states that in Polish spaces Suslin sets (our ω-Suslin sets) are precisely the analytic sets, see e.g. [K, Theorem 25.7].

Some more notation: By $B(x, r)$, resp. $U(x, r)$ we denote the closed, resp. open ball in a metric space centred at x and with radius r. By $\mathcal{L}(X; Y)$, resp. $\mathcal{P}(^nX; Y)$ we denote the space of continuous linear operators, resp. continuous n-homogeneous polynomials from X to Y; for an introduction to the theory of polynomials see e.g. [HJ, Chapter 1]. If f is a mapping into a vector space Y, then $\operatorname{supp}_\circ f = f^{-1}(Y \setminus \{0\})$. If $x, y \in Y$, then $[x, y]$ denotes the segment with endpoints x and y.

Our main result is the following:

Theorem 1. *Let X be an infinite-dimensional Banach space that admits a C^k-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$, with each derivative bounded on X. Let Y be a Banach space with $\operatorname{dens} Y \le \operatorname{dens} X$, let $C \subset Y$ be convex, $y_1 \in C$, and $C \subset A \subset \overline{C}$ a $\operatorname{dens} Y$-Suslin set. Then there is $f \in C^k(X; Y)$ with $\operatorname{supp}_\circ f \subset B_X$ such that $f(X) = [0, y_1] \cup A$.*

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Before proving the theorem we show some of its corollaries. For this we also need the following theorem of Felix Hausdorff, [Ha, Satz I], see also [J, Exercise 29.8].

Theorem 2. *Every uncountable Polish space is a union of an increasing ω_1-sequence of G_δ sets.*

Corollary 3. *Let X be a non-separable Banach space that admits a C^k-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$ with each derivative bounded on X, and let Y be a separable Banach space with* $\dim Y \geq 2$. *Then there is $f \in C^k(X;Y)$ such that $f(X) = Y$ but $f(Z) \neq Y$ for any separable subset $Z \subset X$.*

Proof. Let $g \in Y^*$, $g \neq 0$. Set $A_1 = \{y \in Y;\ g(y) > 0\} \cup \{0\}$ and $C = \{y \in Y;\ g(y) < 0\} \cup \{0\}$, and note that C is convex. Since $\dim Y \geq 2$, $\ker g$ is non-trivial and by Theorem 2 there are G_δ sets $H_\alpha \subset \ker g$, $\alpha \in [2, \omega_1)$ such that $H_\alpha \subsetneq H_\beta$ for $2 \leq \alpha < \beta < \omega_1$ and $\ker g = \bigcup_{\alpha \in [2,\omega_1)} H_\alpha$. Set $A_\alpha = C \cup H_\alpha$ for $\alpha \in [2, \omega_1)$ and note that $\bigcup_{\alpha \in [1,\omega_1)} A_\alpha = Y$, but if β is a countable ordinal, then $\bigcup_{\alpha \in [1,\beta)} A_\alpha \neq Y$. Let $\{B_\alpha \subset X\}_{\alpha \in [1,\omega_1)}$ be a uniformly discrete system of balls of radius 1. By Theorem 1 (using $y_1 = 0$) there are mappings $f_\alpha \in C^k(X;Y)$ such that $\operatorname{supp}_\circ f_\alpha \subset B_\alpha$ and $f_\alpha(X) = A_\alpha$ for each $\alpha \in [1, \omega_1)$. It follows that $f = \sum_{\alpha \in [1,\omega_1)} f_\alpha \in C^k(X;Y)$ and $f(X) = Y$. On the other hand, if $Z \subset X$ is a separable subset, then Z meets at most countably many of the balls B_α and consequently $f(Z) \neq Y$.

□

To relax the assumption on the existence of a suitable bump in the previous corollary we use the following non-separable variant of [Há2, Theorem 4]. (We note that in the separable case the assumption is in particular satisfied if there is a non-compact operator from X into ℓ_p, see [HJ, Proposition 3.33].)

Theorem 4. *Let X be a Banach space for which there is $T \in \mathcal{L}(X;\ell_p(\Gamma))$ for some infinite Γ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis $\{e_\gamma\}_{\gamma\in\Gamma}$. Then for every Banach space Y of density at most* $\operatorname{card}\Gamma$ *there exists a polynomial surjection $P \in \mathcal{P}({}^{\lceil p\rceil}X;Y)$.*

Proof. Let $\{x_\gamma\}_{\gamma\in\Gamma} \subset B_X$ be such that $T(x_\gamma) = e_\gamma$, $\gamma \in \Gamma$, and let $\{y_\gamma\}_{\gamma\in\Gamma}$ be a dense set in B_Y. Denote $m = \lceil p \rceil$ and define $Q\colon \ell_p(\Gamma) \to Y$ by $Q(z) = \sum_{\gamma\in\Gamma} f_\gamma(z)^m y_\gamma$, where f_γ are the canonical coordinate functionals. Then $Q \in \mathcal{P}({}^m\ell_p(\Gamma);Y)$ by [HJ, Theorem 1.29] (consider the net indexed by the directed set of all finite subsets of Γ). Finally, define $P \in \mathcal{P}({}^mX;Y)$ by $P = Q \circ T$. Now if $y \in Y$, then by [HJ, Fact 6.64] there is a sequence $\{\gamma_n\}_{n=0}^\infty$ of distinct elements of Γ such that $y = \|y\| \sum_{n=0}^\infty 2^{-mn} y_{\gamma_n}$. Put $x = \|y\|^{\frac{1}{m}} \sum_{n=0}^\infty 2^{-n} x_{\gamma_n}$. Then $P(x) = Q\bigl(\|y\|^{\frac{1}{m}} \sum_{n=0}^\infty 2^{-n} e_{\gamma_n}\bigr) = \|y\| \sum_{n=0}^\infty 2^{-mn} y_{\gamma_n} = y$. □

The next corollary in particular solves Problem 186 from [GMZ].

Corollary 5. *Let X be a Banach space for which there is $T \in \mathcal{L}(X;\ell_p(\Gamma))$ for some uncountable Γ and $1 \leq p < \infty$ such that $T(B_X)$ contains the canonical basis of $\ell_p(\Gamma)$. (This holds in particular if X has a non-separable super-reflexive quotient.) Then for any separable Banach space Y with* $\dim Y \geq 2$ *there is $f \in C^\infty(X;Y)$ such that $f(X) = Y$ but $f(Z) \neq Y$ for any separable subset $Z \subset X$.*

Proof. By Theorem 4 there is a polynomial surjection $P\colon X \to \ell_2(\Gamma)$. By Corollary 3 there is $g \in C^\infty(\ell_2(\Gamma);Y)$ such that $g(\ell_2(\Gamma)) = Y$ but $g(Z) \neq Y$ for any separable subset $Z \subset \ell_2(\Gamma)$. To finish we set $f = g \circ P$.

If X is non-separable and super-reflexive, then there is a bounded linear injection from X into $\ell_p(\Gamma)$, see e.g. [JTZ, proof of Lemma 2]. The existence of the operator T now follows from Corollary 12 used with $\mu = \omega_1$. As for the quotient, see the remark preceding Corollary 12.

□

We note that for $k > 1$ the assumption of Corollary 3 is stronger than the assumption of Corollary 5. Indeed, if X admits a $C^{1,1}$-smooth bump, then it is already super-reflexive.

We now proceed to prove Theorem 1. This will be done with the help of the next two auxiliary statements.

Lemma 6. *Let X be an infinite-dimensional Banach space that admits a function $\varphi \in C^k(X;[0,1])$, $k \in \mathbb{N} \cup \{\infty\}$, with each derivative bounded on X, and such that* $\operatorname{supp}_\circ \varphi \subset B_X$ *and $\varphi = 1$ on $B(0,r)$ for some $r \in (0,1)$. Let T be a rooted* dens *X-branching tree of height ω. Let $n_0 \in \mathbb{N}$ and $\{\varepsilon_n\}_{n=n_0}^\infty \subset (0,+\infty)$ be such that $\varepsilon_n \to 0$. Let Y be a Banach space and let $\{y_t\}_{t\in T} \subset Y$ be such that $y_0 = 0$ and $\|y_t - y_{t^-}\| \leq \varepsilon_n\bigl((\frac{r}{4})^k\bigr)^n$ for each $t \in T_n$, $n \in \mathbb{N}$, $n \geq n_0$ if $k \in \mathbb{N}$, resp. $\|y_t - y_{t^-}\| \leq \varepsilon_n\bigl((\frac{r}{4})^n\bigr)^n$ if $k = \infty$. Then there is $f \in C^k(X;Y)$ such that* $\operatorname{supp}_\circ f \subset B_X$ *and*

$$f(X) = \bigcup_{t\in T\setminus\{0\}} [y_{t^-}, y_t] \cup \bigl\{\lim_{n\to\infty} y_{b_n};\ b \in \mathcal{B}(T)\bigr\}.$$

Proof. Note that $T_0 = \{0\}$. By induction on the tree levels we find a collection $\{x_t\}_{t\in T} \subset X$ such that

(i) $U\bigl(x_s, (\frac{r}{4})^n\bigr) \subset U\bigl(x_t, r(\frac{r}{4})^{n-1}\bigr)$ for each $n \in \mathbb{N}$, $t \in T_{n-1}$, and $s \in t^+$, and
(ii) for each $n \in \mathbb{N}_0$ the family $\bigl\{U\bigl(x_t, (\frac{r}{4})^n\bigr)\bigr\}_{t\in T_n}$ is uniformly discrete.

Set $x_0 = 0$. Let $n \in \mathbb{N}$ and assume that $\{x_t\}_{t\in T_{n-1}}$ are already defined. By [HJ, Fact 6.65] each ball $B\big(x_t, \frac{3}{4}r(\frac{r}{4})^{n-1}\big)$, $t \in T_{n-1}$ contains a $\frac{2}{3}r(\frac{r}{4})^{n-1}$-separated set $\{x_s\}_{s\in t^+}$. Then $U\big(x_s, \frac{1}{4}r(\frac{r}{4})^{n-1}\big) \subset U\big(x_t, r(\frac{r}{4})^{n-1}\big)$ for each $s \in t^+$ and so (i) holds. Also, each family $\big\{U\big(x_s, (\frac{r}{4})^n\big)\big\}_{s\in t^+}$ is $\frac{1}{6}r(\frac{r}{4})^{n-1}$-uniformly discrete and combining this with (i) and the inductive hypothesis gives (ii).

Next, for $n \in \mathbb{N}$ and $x \in X$ we set

$$f_n(x) = \sum_{i=1}^{n} \sum_{t\in T_i} (y_t - y_{t^-})\varphi\big((\tfrac{4}{r})^i(x - x_t)\big).$$

The inner sum is locally finite by (ii) and hence $f_n \in C^k(X;Y)$ and

$$D^j f_n(x) = \sum_{i=1}^{n} \sum_{t\in T_i} (y_t - y_{t^-})(\tfrac{4}{r})^{ij} D^j\varphi\big((\tfrac{4}{r})^i(x - x_t)\big)$$

for each $x \in X$ and $j < k+1$. In fact, since $D^j\varphi\big((\frac{4}{r})^i(x - x_t)\big)$ is non-zero only for $x \in U\big(x_t, (\frac{r}{4})^i\big) \setminus B\big(x_t, r(\frac{r}{4})^i\big)$, by (i) and (ii) we see that at each $x \in X$ only one summand overall in the formula for $D^j f_n(x)$ can be non-zero and so we have the following estimate:

$$\|D^j f_m(x) - D^j f_l(x)\| = \|D^j(f_m - f_l)(x)\| \le \max_{i=l+1,\dots,m} \varepsilon_i \big((\tfrac{r}{4})^j\big)^i (\tfrac{4}{r})^{ij} C_j \le C_j \sup_{i>l} \varepsilon_i$$

for $x \in X$, $m > l \ge n_0$, $l \ge j$, and $j < k+1$, where $C_j > 0$ is such that $D^j\varphi$ is bounded by C_j. It follows by [HJ, Theorem 1.85] that $f_n \to f \in C^k(X;Y)$ uniformly on X.

Finally, note that $\varphi\big(X \setminus U(0,r)\big) = \varphi\big(B(0,1) \setminus U(0,r)\big) = [0,1]$. Hence, by using induction on n and properties (i) and (ii) we obtain $f_n\big(B\big(x_t, r(\frac{r}{4})^n\big)\big) = y_t$ for each $t \in T_n$ and $f(X \setminus G_n) = f_n(X \setminus G_n) = \bigcup_{1\le i\le n} \bigcup_{t\in T_i} [y_{t^-}, y_t]$, where $G_n = \bigcup_{t\in T_n} U\big(x_t, r(\frac{r}{4})^n\big)$. On the other hand, by (i) and (ii), $x \in \bigcap_{n=1}^\infty G_n$ if and only if there is a branch $b \in \mathcal{B}(T)$ such that $x = \lim_{n\to\infty} x_{b_n}$. It follows that $x \in U\big(x_{b_n}, r(\frac{r}{4})^n\big)$ for each $n \in \mathbb{N}$ and consequently $f_n(x) = y_{b_n}$. Therefore $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} y_{b_n}$.

□

Recall that the Lindelöf number of a topological space is the smallest infinite cardinal number μ such that every open covering of this space has a subcovering of cardinality at most μ. For metric spaces the Lindelöf number is equal to the density.

Proposition 7. *Let (X, ρ) be a metric space with Lindelöf number μ, $U \subset X$, let $A \subset \overline{U}$ be a non-empty μ-Suslin set, and let $\{\varepsilon_n\}_{n=1}^\infty \subset (0, +\infty)$. Then there is a rooted μ-branching tree T of height ω and a family $\{x_t\}_{t\in T} \subset U$ such that $A = \big\{\lim_{n\to\infty} x_{b_n};\ b \in \mathcal{B}(T)\big\}$ and $\rho(x_u, x_t) < \varepsilon_n$ for each $u \in t^+$, $t \in T_n$, $n \in \mathbb{N}$.*

Proof. There is a μ-branching tree S of height ω and closed sets $F_s \subset \overline{U}$, $s \in S$ such that $F_u \subset F_s$ if $u \in s^+$ and $A = \bigcup_{b\in\mathcal{B}(S)} \bigcap_{n=1}^\infty F_{b_n}$. Without loss of generality we may assume that S is rooted and that $\varepsilon_n \to 0$. We construct the tree T and the family $\{x_t\}_{t\in T}$ by induction on the tree level. The members of T will be pairs (s, α), where $s \in S$. We set the least element of T as $(0,0)$ and choose $x_{(0,0)} \in U$ arbitrarily. To start the induction we also put $\varepsilon_0 = +\infty$. Let $n \in \mathbb{N}_0$ and assume that T_n and x_t, $t \in T_n$ are already defined. Fix $t = (s, \alpha) \in T_n$. For each $u \in s^+$ there is a covering of $U(x_t, \varepsilon_n) \cap F_u$ by balls $U(x_{(u,\gamma)}, \varepsilon_{n+1})$, $\gamma \in \Gamma_{\alpha,u}$ such that $x_{(u,\gamma)} \in U \cap U(x_t, \varepsilon_n)$, $\operatorname{dist}(x_{(u,\gamma)}, F_u) < \frac{1}{n+1}$, and $\operatorname{card} \Gamma_{\alpha,u} \le \mu$. Set $t^+ = \{(u,\gamma);\ u \in s^+, \gamma \in \Gamma_{\alpha,u}\}$. Then clearly $\operatorname{card} t^+ \le \mu$ and $\rho(x_v, x_t) < \varepsilon_n$ for any $v \in t^+$.

Now for a given $x \in A$ there is a branch $b \in \mathcal{B}(S)$ such that $x \in \bigcap_{n=1}^\infty F_{b_n}$. By induction it is easy to see that for each $n \in \mathbb{N}$ there is $\beta_n \in \Gamma_{\beta_{n-1}, b_n}$ such that $x \in F_{b_n} \cap U(x_{(b_n,\beta_n)}, \varepsilon_n)$. Put $c_0 = (0,0)$ and $c_n = (b_n, \beta_n)$. Then $\{c_n\}_{n=0}^\infty$ is a branch in T and clearly $\lim_{n\to\infty} x_{c_n} = x$. Thus $A \subset \big\{\lim_{n\to\infty} x_{d_n};\ d \in \mathcal{B}(T)\big\}$.

On the other hand, suppose that $c \in \mathcal{B}(T)$ and $x = \lim_{n\to\infty} x_{c_n}$. Then $c_n = (b_n, \beta_n)$ with $\{b_n\}_{n=0}^\infty$ being a branch in S. Since $\operatorname{dist}(x_{c_n}, F_{b_n}) < \frac{1}{n}$ and $F_{b_n} \subset F_{b_k}$ if $n \ge k$, it follows that $x \in F_{b_k}$ for each $k \in \mathbb{N}$ and consequently $x \in \bigcap_{n=1}^\infty F_{b_n}$. Thus $\big\{\lim_{n\to\infty} x_{d_n};\ d \in \mathcal{B}(T)\big\} \subset A$.

□

Proof of Theorem 1. By composing the bump with a suitable smooth real function we obtain $r \in (0,1)$ and φ as in Lemma 6. Set $\varepsilon_n = \frac{1}{n}\big(\frac{r}{4}\big)^{kn}$ if $k \in \mathbb{N}$, resp. $\varepsilon_n = \frac{1}{n}\big(\frac{r}{4}\big)^{n^2}$ if $k = \infty$. By Proposition 7 there is a rooted dens Y-branching (and hence also dens X-branching) tree T of height ω and a family $\{y_t\}_{t\in T} \subset C$ such that $A = \big\{\lim_{n\to\infty} y_{b_n};\ b \in \mathcal{B}(T)\big\}$ and $\|y_t - y_{t^-}\| < \varepsilon_{n+1}$ for each $t \in T_n$, $n \ge 2$. By relabelling and adding a node we may assume that $T_0 = \{0\}$, $T_1 = \{1\}$, and $y_0 = 0$. By Lemma 6 there is $f \in C^k(X;Y)$ such that $\operatorname{supp}_o f \subset B_X$ and $f(X) = A \cup \bigcup_{t\in T\setminus\{0\}} [y_{t^-}, y_t] = [0, y_1] \cup A$.

□

2. Canonical basis of $\ell_p(\Gamma)$ in a linear image

In this section we look for some sufficient conditions on the space X ensuring that there is a bounded linear operator $T\colon X \to \ell_p(\Gamma)$ such that $T(B_X)$ contains the canonical basis. For $p = 1$ we have the following simple and certainly well-known observation.

Fact 8. *Let X be a Banach space. Then there is $T \in \mathcal{L}(X;\ell_1(\Gamma))$ such that $T(B_X)$ contains the canonical basis $\{e_\gamma\}_{\gamma\in\Gamma}$ if and only if X has a complemented subspace isomorphic to $\ell_1(\Gamma)$.*

Proof. $\Leftarrow$ We can take the projection composed with the isomorphism and suitably scaled.

$\Rightarrow$ We can use the lifting property of $\ell_1(\Gamma)$ once we realise that T is in fact onto. Indeed, let $x_\gamma \in B_X$ be such that $T(x_\gamma) = e_\gamma$. Any $y \in \ell_1(\Gamma)$ is of the form $y = \sum_{n=1}^\infty a_n e_{\gamma_n}$. We can put $x = \sum_{n=1}^\infty a_n x_{\gamma_n}$, since the series converges absolutely. Then $T(x) = y$.

□

Now we give several auxiliary technical statements leading finally to the conditions given in Corollary 12.

Let X be a normed linear space and $M \subset X^*$. We say that a net $\{x_\alpha\}_{\alpha\in\Gamma} \subset X$ is M-null if $\lim_\alpha f(x_\alpha) = 0$ for each $f \in M$. Recall that the cofinality of an infinite cardinal μ, denoted by $\operatorname{cf}\mu$, is the smallest cardinal ν such that $[0,\mu)$ has a subset A of cardinality ν with $\sup A = \mu$; μ is called regular if $\operatorname{cf}\mu = \mu$.

Lemma 9. *Let X be a normed linear space and $\{f_\gamma\}_{\gamma\in\Gamma} \subset X^*$. For $x \in X$ we denote $\operatorname{supp} x = \{\gamma \in \Gamma;\ f_\gamma(x) \neq 0\}$. Let $\mu > \omega$ be a regular cardinal and $\{x_\alpha\}_{\alpha\in[0,\mu)} \subset X$ an $\{f_\gamma\}$-null net such that $\operatorname{card}\operatorname{supp} x_\alpha < \mu$ for each $\alpha \in [0,\mu)$. Then there is a subnet $\{y_\alpha\}_{\alpha\in[0,\mu)}$ of $\{x_\alpha\}_{\alpha\in[0,\mu)}$ with disjoint supports, i.e. $\operatorname{supp} y_\alpha \cap \operatorname{supp} y_\beta = \emptyset$ for any $\alpha, \beta \in [0,\mu)$, $\alpha \neq \beta$.*

Proof. Since $\operatorname{cf}\mu > \omega$, for each $\gamma \in \Gamma$ there is $G(\gamma) \in [0,\mu)$ such that $f_\gamma(x_\alpha) = 0$ for $\alpha \in [G(\gamma),\mu)$. We define an increasing $F\colon [0,\mu) \to [0,\mu)$ such that $\operatorname{supp} x_{F(\alpha)} \cap \operatorname{supp} x_{F(\beta)} = \emptyset$ whenever $0 \leq \alpha < \beta < \mu$ by transfinite recursion. Put $F(0) = 0$. Let $\beta \in (0,\mu)$ and put $\Lambda = \bigcup_{\alpha\in[0,\beta)} \operatorname{supp} x_{F(\alpha)}$. Then $\operatorname{card}\Lambda < \mu$ since μ is regular. Put $\eta = \sup_{\gamma\in\Lambda} G(\gamma)$. Then again $\eta < \mu$ by the regularity. We set $F(\beta) = \max\bigl\{\eta, \sup_{\alpha\in[0,\beta)}(F(\alpha)+1)\bigr\}$ and note that $f_\gamma(x_{F(\beta)}) = 0$ for $\gamma \in \Lambda$.

Finally, we put $y_\alpha = x_{F(\alpha)}$ for $\alpha \in [0,\mu)$, which clearly defines a (Willard) subnet of $\{x_\alpha\}_{\alpha\in[0,\mu)}$.

□

Proposition 10. *Let X be a normed linear space and $T \in \mathcal{L}(X;\ell_p(\Gamma))$ for some Γ and $1 \leq p < \infty$. Let $\mu > \omega$ be a regular cardinal. Then for each weakly null net $\{x_\alpha\}_{\alpha\in[0,\mu)} \subset X \setminus \ker T$ there is a subnet $\{y_\alpha\}_{\alpha\in[0,\mu)}$ and $S \in \mathcal{L}\bigl(X;\ell_p([0,\mu))\bigr)$ such that $S(y_\alpha) = e_\alpha$, $\alpha \in [0,\mu)$, where $\{e_\alpha\}_{\alpha\in[0,\mu)}$ is the canonical basis of $\ell_p([0,\mu))$.*

The same holds if we consider $c_0(\Gamma)$ and $c_0([0,\mu))$ instead of $\ell_p(\Gamma)$ and $\ell_p([0,\mu))$.

Proof. Consider the sets $\Gamma_n = \bigl\{\alpha \in [0,\mu);\ \frac{1}{n} \leq \|T(x_\alpha)\| \leq n\bigr\}$. Since $\operatorname{cf}\mu > \omega$, there is $n \in \mathbb{N}$ such that $\operatorname{card}\Gamma_n = \mu$. Thus by passing to a subnet we may assume that $\{T(x_\alpha)\}_{\alpha\in[0,\mu)}$ is semi-normalised. Since T is w–w continuous, $\{T(x_\alpha)\}_{\alpha\in[0,\mu)}$ is weakly null. By Lemma 9 there is a subnet $\{y_\alpha\}_{\alpha\in[0,\mu)}$ of $\{x_\alpha\}_{\alpha\in[0,\mu)}$ such that $\{T(y_\alpha)\}_{\alpha\in[0,\mu)}$ have disjoint supports. Consequently there is a bounded linear projection $P\colon \ell_p(\Gamma) \to \overline{\operatorname{span}}\{T(y_\alpha)\}$ and an isomorphism $R \in \mathcal{L}\bigl(\overline{\operatorname{span}}\{T(y_\alpha)\};\ell_p([0,\mu))\bigr)$ with $R(T(y_\alpha)) = e_\alpha$. We may then set $S = R \circ P \circ T$.

□

Let X be an infinite-dimensional normed linear space and μ a cardinal. For an application of Proposition 10 we need to find a non-trivial weakly null long sequence in X. Notice that if $M \subset X^*$ separates the points of X, then there is no $\sigma(X,M)$-null long sequence of length μ in $X \setminus \{0\}$ when $\operatorname{cf}\mu > \operatorname{card} M$. Indeed, M gives rise to a neighbourhood basis of $\sigma(X,M)$ of cardinality $\operatorname{card} M$ and thus any $\sigma(X,M)$-null long sequence of length μ is eventually zero. Consequently, there is no weakly null long sequence of length μ in $X \setminus \{0\}$ when $\operatorname{cf}\mu > w^*\text{-}\operatorname{dens} X^*$ and there is no non-zero w^*-null long sequence of length ω_1 in ℓ_∞. In particular, if we want to have a weakly null long sequence of length $\operatorname{dens} X$ in $X \setminus \{0\}$ with $\operatorname{dens} X$ a regular cardinal, then the space X has to be a DENS space, i.e. a space for which $w^*\text{-}\operatorname{dens} X^* = \operatorname{dens} X$.

Recall that a Banach space X is weakly Lindelöf determined (WLD) if and only if there is a one-to-one w^*–pointwise continuous bounded linear operator $T\colon X^* \to \ell_\infty^c(\Gamma)$ for some set Γ, see [AM]. Clearly, a quotient of a WLD space is again WLD. Note also that a WLD space is a DENS space, [HMVZ, Proposition 5.40].

Lemma 11. *Let X be a WLD Banach space and $\mu \leq \operatorname{dens} X$ a cardinal with $\operatorname{cf}\mu > \omega$. Then there is a normalised uniformly separated weakly null net $\{x_\alpha\}_{\alpha\in[0,\mu)} \subset X$.*

Proof. There is a Markushevich basis $\{(x_\gamma; f_\gamma)\}_{\gamma\in\Gamma}$ of X that countably supports X^*, i.e. the set $\{\gamma \in \Gamma;\ f(x_\gamma) \neq 0\}$ is countable for every $f \in X^*$, see e.g. [HMVZ, Theorem 5.37]. It follows that $\{x_{F(\alpha)}\}_{\alpha\in[0,\mu)}$ is weakly null for any one-to-one mapping $F\colon [0,\mu) \to \Gamma$. Now consider the sets $\Gamma_n = \bigl\{\gamma \in \Gamma;\ \|x_\gamma\| \leq n, \|f_\gamma\| \leq n\bigr\}$. Then $\operatorname{card}\Gamma_n \geq \mu$ for some $n \in \mathbb{N}$. Also, $\|x_\alpha - x_\beta\| \geq \frac{1}{\|f_\alpha\|}|f_\alpha(x_\alpha - x_\beta)| = \frac{1}{\|f_\alpha\|} \geq \frac{1}{n}$ for $\alpha, \beta \in \Gamma_n$, $\alpha \neq \beta$. Hence the set $\{x_\gamma\}_{\gamma\in\Gamma_n}$ is bounded and has a positive distance from the origin. Thus we may put $y_\gamma = \frac{x_\gamma}{\|x_\gamma\|}$ for $\gamma \in \Gamma_n$, and $\{y_\gamma\}_{\gamma\in\Gamma_n}$ satisfies the requirements.

□

Note that if X, Y are Banach spaces and $Q \in \mathcal{L}(X;Y)$ is onto, then by the open mapping theorem after scaling Q we obtain $R \in \mathcal{L}(X;Y)$ such that $B_Y \subset R(B_X)$. Now if $T \in \mathcal{L}(Y;\ell_p(\Gamma))$ is such that $T(B_Y)$ contains the canonical basis of $\ell_p(\Gamma)$, then $T \circ R\colon X \to \ell_p(\Gamma)$ has the same property.

Corollary 12. *Let X be a Banach space, $\mu > \omega$ a regular cardinal, Γ a set, and $1 < p < \infty$. Consider the following conditions:*

(i) X is WLD and there is $T \in \mathcal{L}(X;\ell_p(\Gamma))$ such that $\operatorname{dens} X/\ker T \geq \mu$.

(ii) X contains a non-zero weakly null net $\{x_\alpha\}_{\alpha\in[0,\mu)}$ and there is $T \in \mathcal{L}(X;\ell_p(\Gamma))$ such that $\operatorname{dens}\ker T < \mu$.

If one of the above conditions is satisfied, then there is $S \in \mathcal{L}\big(X; \ell_p([0,\mu))\big)$ *such that* $S(B_X)$ *contains the canonical basis of* $\ell_p([0,\mu))$.

Proof. (i) Let $Z = X/\ker T$. According to the remark preceding Corollary 12 it suffices to find the required operator from Z. Let $Q\colon X \to Z$ be the canonical quotient mapping. Define $\widehat{T}\colon Z \to \ell_p(\Gamma)$ by $\widehat{T}(z) = T(x)$ for some $x \in Q^{-1}(z)$. Then $\widehat{T} \in \mathcal{L}(Z; \ell_p(\Gamma))$ and it is one-to-one. Also, Z is WLD with dens $Z \geq \mu$ and hence by combining Lemma 11 and Proposition 10 there exists $S \in \mathcal{L}\big(Z; \ell_p([0,\mu))\big)$ such that $S(B_Z)$ contains the canonical basis of $\ell_p([0,\mu))$.

(ii) Since $\operatorname{cf}\mu > \omega$, we may assume without loss of generality that $\{x_\alpha\}_{\alpha\in[0,\mu)}$ is semi-normalised and contained in B_X. Also, by [T, Theorem 1.1] we may assume that $\{x_\alpha\}_{\alpha\in[0,\mu)}$ is a long Schauder basic sequence, and in particular that it is uniformly separated. Consequently there is $\beta < \mu$ such that $x_\alpha \notin \ker T$ for $\alpha \geq \beta$ and we may apply Proposition 10.

□

Finally, note that ℓ_∞ has a quotient isomorphic to $\ell_2(\mathfrak{c})$ ([HMVZ, Theorem 4.22]) although it does not contain a non-zero w^*-null long sequence of length ω_1.

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