SMOOTH PARTITIONS OF UNITY ON BANACH SPACES

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ABSTRACT. We show a new characterisation of the existence of smooth partitions of unity on a Banach space. This leads to a slight generalisation of some as well as a very easy recovery of most of the known results using a unified treatment.

Smooth partitions of unity are an important tool in the theory of smooth approximations (see [HJ, Chapter 7]), smooth extensions, theory of manifolds, and other areas. Clearly a necessary condition for a Banach space to admit smooth partitions of unity is the existence of a smooth bump function. The sufficiency of this condition for a general Banach space is still an open problem. A positive answer was established in many cases, the most important of which are the following (i.e. if one of the conditions below is fulfilled, then the existence of a smooth bump function on X implies that X admits smooth partitions of unity):

(i) X has an SPRI (separable "projectional resolution of the identity"), [GTWZ].
(ii) X belongs to a $\overline{\mathcal{P}}$-class, [H].
(iii) $X = C(K)$ for K compact, [HH].
(iv) X has a subspace Y isomorphic to $c_0(\Gamma)$ such that X/Y admits smooth partitions of unity, [DGZ1].
(v) X^* is weakly compactly generated (WCG), [M].

For the definition and basic properties of an SPRI see [F, Definition 6.2.6 ff.] or [HMVZ, Theorem 3.46]; for the definition of a $\overline{\mathcal{P}}$-class see p. 3.

The original proofs of the results (i), (iv), and (v) use Toruńczyk's characterisation of the existence of smooth partitions of unity by non-linear homeomorphic embedding into $c_0(\Gamma)$ with smooth component functions (see e.g. [HJ, Proposition 7.60]). The other two results use the following theorem of Richard Haydon:

Theorem 1 ([H], see also [HJ, Theorem 7.53])**.** *Let X be a normed linear space that admits a C^k-smooth bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Γ be a set and $\Phi\colon X \to c_0(\Gamma)$ a continuous mapping such that for every $\gamma \in \Gamma$ the function $e_\gamma^* \circ \Phi$ is C^k-smooth. For each finite $F \subset \Gamma$ let $P_F \in C^k(X;X)$ be such that the space* $\operatorname{span} P_F(X)$ *admits locally finite $\mathcal{C}^k$-partitions of unity. Assume that for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - P_F(x)\| < \varepsilon$ if we set $F = \{\gamma \in \Gamma;\ |\Phi(x)(\gamma)| \geq \delta\}$. Then X admits locally finite and σ-uniformly discrete $\mathcal{C}^k$-partitions of unity.*

While pondering the applicability of Haydon's theorem we were led to another characterisation of the existence of smooth partitions of unity (Theorem 2). This characterisation allows very easy recovery of all the results above except for the $C(K)$ case. In fact, an immediate consequence is a (at least formal) generalisation of (i), (ii), and (v) given in Corollary 6, which puts all these results under a common roof (this is either obvious or shown in Theorem 10 and Corollary 9). There is also another tiny advantage for the insight into the problem when using Theorem 1: All the original proofs that use Toruńczyk's characterisation (of course they all come from the same workshop) at some point invoke the completeness of the underlying space, but as we shall see here, the completeness is completely irrelevant to the problem.

Before we start, we fix some notation. By $U(x,r)$, resp. $B(x,r)$ we denote the open, resp. closed ball centred at x with radius r. For a function $f\colon X \to \mathbb{R}$ we denote $\operatorname{supp}_o f = f^{-1}(\mathbb{R} \setminus \{0\})$. For other unfamiliar notation or terminology see [HJ] or [FHHMZ].

Now, the reason that Haydon's theorem can be successfully used to prove the wonderful result (iii) is that there is a rich supply of projections of norm one on an Asplund $C(K)$ space (formed by restrictions to clopen subsets of K). So what do we have on an arbitrary Banach space? The projections onto one-dimensional subspaces, of course. This observation leads to the following characterisation:

Theorem 2. *Let X be a normed linear space and $k \in \mathbb{N} \cup \{\infty\}$. The following statements are equivalent:*

(i) X admits locally finite and σ-uniformly discrete $\mathcal{C}^k$-partitions of unity.
(ii) X admits a C^k-smooth bump and there are a set Γ, a continuous $\Phi\colon X \to c_0(\Gamma)$ such that $e_\gamma^* \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$, and $\{x_{\gamma n}\}_{\gamma\in\Gamma, n\in\mathbb{N}} \subset X$ such that $x \in \overline{\operatorname{span}}\{x_{\gamma n};\ \Phi(x)(\gamma) \neq 0, n \in \mathbb{N}\}$ for every $x \in X$.

Notice that the condition (ii) resembles a property of a strong Markushevich basis.

Before delving into the proof of Theorem 2 we make a short technical intermission. Applications of Theorem 1 involve constructions of continuous mappings into $c_0(\Gamma)$. To avoid repeating the same argument in several of these constructions we will make use of the following simple lemma.

Date: October 2016.
2010 *Mathematics Subject Classification.* 46B20, 46B26, 46T20.
Key words and phrases. Smooth partitions of unity.
Supported by GAČR 16-07378S.

Lemma 3. *Let X be a topological space, Γ a set, and $\Phi\colon X \to \mathbb{R}^{\mathbb{N}\times\Gamma}$. Suppose that all the component functions $x \mapsto \Phi(x)(n,\gamma)$ are continuous, $\lim_{n\to\infty}\Phi(x)(n,\gamma) = 0$ locally uniformly in $x \in X$ and uniformly in $\gamma \in \Gamma$, and for each fixed $n \in \mathbb{N}$, $x \in X$, and $\varepsilon > 0$ there are a neighbourhood U of x and a finite $F \subset \Gamma$ such that $|\Phi(y)(n,\gamma)| < \varepsilon$ whenever $y \in U$ and $\gamma \in \Gamma \setminus F$. Then Φ is a continuous mapping into $c_0(\mathbb{N}\times\Gamma)$.*

Proof. Fix $x \in X$ and $\varepsilon > 0$. There are $n_0 \in \mathbb{N}$ and a neighbourhood U of x such that $|\Phi(y)(n,\gamma)| < \frac{\varepsilon}{2}$ whenever $n > n_0$, $y \in U$, and $\gamma \in \Gamma$. For each $n \in \mathbb{N}$, $n \le n_0$ there are a neighbourhood $V_n \subset U$ of x and a finite $F_n \subset \Gamma$ such that $|\Phi(y)(n,\gamma)| < \frac{\varepsilon}{2}$ whenever $y \in V_n$ and $\gamma \in \Gamma \setminus F_n$. Put $F = \bigcup_{n\le n_0}\{n\}\times F_n$ and $V = \bigcap_{n\le n_0} V_n$. Then F is finite and $|\Phi(y)(n,\gamma)| < \frac{\varepsilon}{2}$ whenever $y \in V$ and $(n,\gamma) \in \mathbb{N}\times\Gamma \setminus F$. This shows that Φ maps into $c_0(\mathbb{N}\times\Gamma)$. The continuity of Φ follows from the fact that $|\Phi(y)(n,\gamma) - \Phi(x)(n,\gamma)| < \varepsilon$ whenever $y \in V$ and $(n,\gamma) \in \mathbb{N}\times\Gamma\setminus F$, and from the continuity of the functions $y \mapsto \Phi(y)(n,\gamma)$, $(n,\gamma) \in F$.

□

Proof of Theorem 2. For the purpose of the proof let us consider the following intermediate statement:

(ii)' X admits a C^k-smooth bump and there are a set Λ, a continuous $\Psi\colon X \to c_0(\Lambda)$ such that $e^*_\lambda \circ \Psi \in C^k(X)$ for every $\lambda \in \Lambda$, and $\{x_\lambda\}_{\lambda\in\Lambda} \subset X$ such that $x \in \overline{\{x_\lambda;\ \Psi(x)(\lambda) \ne 0\}}$ for every $x \in X$.

(ii)'⇒(i) By the assumption there are functions $h_n \in C^k(X;[0,1])$ such that $\operatorname{supp}_\circ h_n \subset U(0,\frac{1}{n})$ and $h_n(0) > 0$. Set $\Gamma = \mathbb{N}\times\Lambda$ and define $\Phi\colon X \to \ell_\infty(\Gamma)$ by

$$\Phi(x)(n,\lambda) = \frac{1}{n}h_n(x - x_\lambda)\Psi(x)(\lambda).$$

Then Φ is a continuous mapping into $c_0(\Gamma)$ by Lemma 3. Clearly, $e^*_{(n,\lambda)} \circ \Phi \in C^k(X)$ for each $(n,\lambda) \in \Gamma$. Next, for each finite non-empty $F \subset \Gamma$ set $m(F) = \max\{n \in \mathbb{N};\ (n,\lambda) \in F \text{ for some } \lambda \in \Lambda\}$, let $\alpha(F) \in \Lambda$ be chosen arbitrarily such that $\bigl(m(F),\alpha(F)\bigr) \in F$, and let $P_F\colon X \to X$ be the linear projection onto $\operatorname{span}\{x_{\alpha(F)}\}$ of norm at most one. We also set $P_\emptyset = 0$. We show that the assumptions of Theorem 1 are satisfied. Each one-dimensional subspace of X admits locally finite $\mathcal{C}^k$-partitions of unity ([HJ, Corollary 7.50]). Given $x \in X$ and $\varepsilon > 0$ find $m \in \mathbb{N}$ such that $\frac{1}{m} \le \frac{\varepsilon}{2}$. By the assumption there is $\alpha \in \Lambda$ such that $\Psi(x)(\alpha) \ne 0$ and x_α is so close to x that $h_m(x - x_\alpha) > 0$. If we set $\delta = |\Phi(x)(m,\alpha)|$ and $F = \{(n,\lambda) \in \Gamma;\ |\Phi(x)(n,\lambda)| \ge \delta\}$, then $(m,\alpha) \in F$ and hence $m(F) \ge m$. Further, $\bigl|\Phi(x)\bigl(m(F),\alpha(F)\bigr)\bigr| \ge \delta > 0$, and in particular $h_{m(F)}(x - x_{\alpha(F)}) > 0$. It follows that $\|x - x_{\alpha(F)}\| < \frac{1}{m(F)} \le \frac{1}{m} \le \frac{\varepsilon}{2}$. Note that $P_F(x_{\alpha(F)}) = x_{\alpha(F)}$ and therefore $\|x - P_F(x)\| \le \|x - x_{\alpha(F)}\| + \|P_F(x_{\alpha(F)}) - P_F(x)\| < \varepsilon$.

(ii)⇒(ii)' Put $\Lambda = c_{00}^{\mathbb{Q}}(\Gamma\times\mathbb{N})$, i.e. the set of all vectors in $c_{00}(\Gamma\times\mathbb{N})$ with rational coordinates. For each $\lambda \in \Lambda$ set $x_\lambda = \sum_{\gamma\in\Gamma,n\in\mathbb{N}}\lambda(\gamma,n)x_{\gamma n}$. Clearly, $\{x_\lambda;\ \lambda\in\Lambda\} = \operatorname{span}_{\mathbb{Q}}\{x_{\gamma n};\ \gamma\in\Gamma, n\in\mathbb{N}\}$. Further, let $q\colon \mathbb{Q}\to\mathbb{N}$ be some one-to-one mapping with $q(0) = 1$ and put $m(\lambda) = \max\{n \in \mathbb{N};\ \lambda(\gamma,n) \ne 0 \text{ for some } \gamma \in \Gamma\}$ for $\lambda \in \Lambda\setminus\{0\}$ and $m(0) = 1$. Finally, define $\Psi\colon X \to \mathbb{R}^\Lambda$ by

$$\Psi(x)(\lambda) = \frac{1}{m(\lambda)\prod_{\gamma\in\Gamma,n\in\mathbb{N}} q\bigl(\lambda(\gamma,n)\bigr)}\prod_{\gamma\in\Gamma\colon \exists n,\,\lambda(\gamma,n)\ne 0}\Phi(x)(\gamma).$$

We claim that Ψ is actually a continuous mapping into $c_0(\Lambda)$.

Indeed, fix $x \in X$ and $\varepsilon > 0$. Since Φ is continuous, there are a neighbourhood U of x and a finite set $H \subset \Gamma$ such that $\|\Phi(y)\| < \|\Phi(x)\| + 1$ and $|\Phi(y)(\gamma)| < 1$ for each $y \in U$ and $\gamma \in \Gamma\setminus H$. Note that $\prod_{\gamma\in\Gamma\colon\exists n,\,\lambda(\gamma,n)\ne0}|\Phi(y)(\gamma)| \le (\|\Phi(x)\|+1)^{|H|}$ for any $y \in U$ and $\lambda \in \Lambda$, and the same holds if we omit any one of the factors in the product. Next, there are a neighbourhood V of x, $V \subset U$, and a finite set $E \subset \Gamma$ such that $|\Phi(y)(\gamma)| < \varepsilon/(\|\Phi(x)\|+1)^{|H|}$ for each $y \in V$ and $\gamma \in \Gamma\setminus E$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \varepsilon/(\|\Phi(x)\|+1)^{|H|}$. Put

$$F = \bigl\{\lambda \in \Lambda;\ \operatorname{supp}\lambda \subset E\times\{1,\dots,N\} \text{ and } q\bigl(\lambda(\gamma,n)\bigr) \le N \text{ for all } \gamma\in\Gamma, n\in\mathbb{N}\bigr\}$$

and note that F is finite. Now if $y \in V$ and $\lambda \in \Lambda\setminus F$, then $|\Psi(y)(\lambda)| < \varepsilon$. It easily follows that Ψ is a continuous mapping into $c_0(\Lambda)$.

Clearly, $e^*_\lambda \circ \Psi \in C^k(X)$ for every $\lambda \in \Lambda$. Finally, given $x \in X$ and a neighbourhood U of x, by the assumption there is $\lambda \in \Lambda$ such that $x_\lambda \in U$ and $\Phi(x)(\gamma) \ne 0$ if $\lambda(\gamma,n) \ne 0$ for some $n \in \mathbb{N}$. Consequently, $\Psi(x)(\lambda) \ne 0$.

(i)⇒(ii) The existence of a C^k-smooth bump is clear (just take a partition of unity subordinated to a covering of X by $U(0,2)$ and $X \setminus B(0,1)$). Next, for each $n \in \mathbb{N}$ let $\{\varphi_{n\lambda}\}_{\lambda\in\Lambda}$ be a locally finite $\mathcal{C}^k$-partition of unity on X subordinated to the uniform covering of X by open balls of radius $\frac{1}{n}$ (clearly $\{\varphi_{n\lambda}\}$ can be constructed by scaling the domains of $\{\varphi_{1\lambda}\}$ so that the index set is always the same). Without loss of generality we may assume that all the functions $\varphi_{n\lambda}$ are non-zero. We put $\Gamma = \mathbb{N}\times\Lambda$ and define $\Phi\colon X \to \ell_\infty(\Gamma)$ by

$$\Phi(x)(n,\lambda) = \frac{1}{n}\varphi_{n\lambda}(x).$$

Then Φ is a continuous mapping into $c_0(\Gamma)$ by Lemma 3. To finish, choose any $x_{n\lambda}$ in each $\operatorname{supp}_\circ\varphi_{n\lambda}$. Fix $x \in X$ and $\delta > 0$. Let $n \in \mathbb{N}$ be such that $\frac{2}{n} < \delta$. There is $\lambda \in \Lambda$ such that $x \in \operatorname{supp}_\circ\varphi_{n\lambda}$. Then $\Phi(x)(n,\lambda) > 0$ and $\|x - x_{n\lambda}\| < \frac{2}{n} < \delta$. It follows that $x \in \overline{\{x_{n\lambda};\ \Phi(x)(n,\lambda) \ne 0\}}$.

□

As a first application we show how the above characterisation can be used to rather easily obtain the result (iv) from the introduction. Not only that our proof is substantially shorter than the original, but it also does not use any fancy tools like lifting, Bartle-Graves selectors, etc. The stripped-down proof clearly exposes the three main ideas behind it: the use of *linear* functionals on the subspace Y, so that they can be extended to the whole space; the use of a fundamental biorthogonal system in Y, which allows to link these extensions to functionals on X/Y; and the crucial property of the norm on $c_0(\Gamma)$: if we drop *all* small coordinates, the vector stays close.

Corollary 4 ([DGZ1])**.** *Let X be a normed linear space and $Y \subset X$ a subspace isomorphic to $c_0(\Gamma)$ for some Γ. If X/Y admits locally finite $\mathcal{C}^k$-partitions of unity for some $k \in \mathbb{N} \cup \{\infty\}$, then X admits locally finite and σ-uniformly discrete $\mathcal{C}^k$-partitions of unity.*

Proof. By extending the equivalent norm from Y we may assume without loss of generality that Y is actually isometric to $c_0(\Gamma)$. Let $Q\colon X \to X/Y$ be the canonical quotient mapping. Let $\{(e_\gamma; f_\gamma)\}_{\gamma\in\Gamma}$ be the canonical basis of $c_0(\Gamma)$ and further assume that each f_γ is actually a norm-one functional on X (use the Hahn-Banach theorem). For each $n \in \mathbb{N}$ let $\{\psi_{n\lambda}\}_{\lambda\in\Lambda}$ be a locally finite $\mathcal{C}^k$-partition of unity on X/Y subordinated to the uniform covering of X/Y by open balls of radius $\frac{1}{n}$ (clearly $\{\psi_{n\lambda}\}$ can be constructed by scaling the domains of $\{\psi_{1\lambda}\}$ so that the index set is always the same). Without loss of generality we may assume that all the functions $\psi_{n\lambda}$ are non-zero. Choose $z_{n\lambda} \in \operatorname{supp}_o \psi_{n\lambda}$ and $x_{n\lambda} \in X$ such that $Q(x_{n\lambda}) = z_{n\lambda}$. Let $\theta_n \in C^\infty\bigl(\mathbb{R}; [0, \frac{1}{n}]\bigr)$, $n \in \mathbb{N}$ be Lipschitz functions satisfying $\theta_n(t) = 0$ if and only if $|t| \le \frac{2}{n}$. We define a mapping $\Phi\colon X \to \ell_\infty(\mathbb{N}\times\Lambda\times\Gamma \cup \mathbb{N}\times\Lambda)$ by

$$\Phi(x)(n,\lambda,\gamma) = \theta_n\bigl(f_\gamma(x - x_{n\lambda})\bigr)\psi_{n\lambda}\bigl(Q(x)\bigr),$$
$$\Phi(x)(n,\lambda) = \frac{1}{n}\psi_{n\lambda}\bigl(Q(x)\bigr).$$

First we show that Φ is actually a continuous mapping into $c_0(\mathbb{N}\times\Lambda\times\Gamma \cup \mathbb{N}\times\Lambda)$. Fix $x \in X$ and $\varepsilon > 0$. Clearly, $0 \le \Phi(y)(n,\lambda,\gamma) < \varepsilon$ and $0 \le \Phi(y)(n,\lambda) < \varepsilon$ for $n > \frac{1}{\varepsilon}$ and all $\lambda \in \Lambda$, $\gamma \in \Gamma$, $y \in X$. Now fix $n \in [1, \frac{1}{\varepsilon}]$. Since $\{\psi_{n\lambda}\}_{\lambda\in\Lambda}$ is locally finite, there is a neighbourhood V of $Q(x)$ and a finite $F \subset \Lambda$ such that $\psi_{n\lambda} = 0$ on V for $\lambda \in \Lambda \setminus F$. Further, there is a neighbourhood U of x such that $Q(U) \subset V$. Then clearly $\Phi(y)(n,\lambda,\gamma) = \Phi(y)(n,\lambda) = 0$ for $y \in U$ and $\lambda \in \Lambda\setminus F$, $\gamma \in \Gamma$. Now fix $\lambda \in F$. Assume that $\psi_{n\lambda}(Q(x)) \ne 0$. Then $\|Q(x) - z_{n\lambda}\| < \frac{2}{n}$. Put $H = \{\gamma \in \Gamma;\ |f_\gamma(x - x_{n\lambda})| \ge \frac{2}{n}\}$. We claim that H is finite. Indeed, if H is infinite, then by the w^*-compactness there is a w^*-accumulation point $f \in B_{X^*}$ of $\{f_\gamma\}_{\gamma\in H}$. Then $f\restriction_Y = 0$, since $\{(e_\gamma; f_\gamma)\}$ is a fundamental biorthogonal system in Y. In particular, f can be considered also as a member of $(X/Y)^*$, and so $\frac{2}{n} \le |f(x - x_{n\lambda})| = |f(Q(x) - z_{n\lambda})| \le \|Q(x) - z_{n\lambda}\| < \frac{2}{n}$, a contradiction. Thus $\Phi(x)(n,\lambda,\gamma) = 0$ for $\gamma \in \Gamma\setminus H$. Since the family of functions $y \mapsto \Phi(y)(n,\lambda,\gamma)$, $\gamma \in \Gamma$, is equi-continuous, there is a neighbourhood W of x such that $|\Phi(y)(n,\lambda,\gamma)| < \varepsilon$ whenever $y \in W$ and $\gamma \in \Gamma\setminus H$. Thus we may apply Lemma 3.

Next, we set $x_{n\lambda\gamma} = e_\gamma$. Fix any $x \in X$ and $\varepsilon > 0$. There is $n \in \mathbb{N}$, $n \ge \frac{8}{\varepsilon}$, and $\lambda \in \Lambda$ such that $\psi_{n\lambda}(Q(x)) > 0$ and $\|Q(x) - z_{n\lambda}\| < \frac{\varepsilon}{4}$. Thus there is $u \in Y$ such that $\|x - x_{n\lambda} - u\| < \frac{\varepsilon}{4}$. Put $F = \{\gamma \in \Gamma;\ |f_\gamma(u)| > \frac{\varepsilon}{2}\}$, which is a finite set (possibly empty), and $v = \sum_{\gamma\in F} f_\gamma(u)e_\gamma$. Then $\|u - v\| \le \frac{\varepsilon}{2}$ (we have the supremum norm here) and so $\|x - (x_{n\lambda} + v)\| \le \|x - x_{n\lambda} - u\| + \|u - v\| < \varepsilon$. Note that $|f_\gamma(x - x_{n\lambda})| \ge |f_\gamma(u)| - |f_\gamma(x - x_{n\lambda} - u)| > \frac{\varepsilon}{2} - \|x - x_{n\lambda} - u\| > \frac{\varepsilon}{4} \ge \frac{2}{n}$ for $\gamma \in F$. Consequently, $\Phi(x)(n,\lambda,\gamma) > 0$ for $\gamma \in F$. It follows that $x \in \overline{\operatorname{span}}\bigl(\{x_{n\lambda};\ \Phi(x)(n,\lambda) \ne 0\} \cup \{x_{n\lambda\gamma};\ \Phi(x)(n,\lambda,\gamma) \ne 0\}\bigr)$.

Each component of Φ is clearly C^k-smooth and the space X admits a C^k-smooth bump by [DGZ1, Proposition 1]. Thus we may conclude the proof by using Theorem 2.

□

Before going further, we review some notions useful in the study of the (linear) structure of non-separable Banach spaces. Let $\mathcal{X}$ be a class of Banach spaces. We say that $\mathcal{X}$ is a $\mathcal{P}$-class if for every non-separable $X \in \mathcal{X}$ there exists a projectional resolution of the identity $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ on X such that $(P_{\alpha+1} - P_\alpha)(X) \in \mathcal{X}$ for all $\alpha < \mu$. We say that $\mathcal{X}$ is a $\overline{\mathcal{P}}$-class if for every non-separable $X \in \mathcal{X}$ there exists a projectional resolution of the identity $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ on X such that $P_\alpha(X) \in \mathcal{X}$ for all $\alpha < \mu$. Note that if a class $\mathcal{X}$ admits PRI and is closed under complemented subspaces, then $\mathcal{X}$ is both $\mathcal{P}$-class and $\overline{\mathcal{P}}$-class. Therefore reflexive, WCG, WCD, and WLD are all both $\mathcal{P}$-classes and $\overline{\mathcal{P}}$-classes, as are 1-Plichko spaces ([HMVZ, Theorem 5.63]; proof of [KKL, Theorem 17.6] combined with [KKL, Theorem 17.16]), spaces with a 1-projectional skeleton (Ondřej Kalenda, private communication; [KKL, Theorem 17.6]), and duals of Asplund spaces ([DGZ, Remark VI.3.5]). Recall that any space from a $\mathcal{P}$-class has an SPRI ([HMVZ, Theorem 3.46]), and any space with an SPRI has a strong Markushevich basis (folklore, see also [HMVZ, Theorem 5.1] or the proof of Theorem 8).

Although the characterisations of the existence of smooth partitions of unity are inherently non-linear, in all the results from the introduction, except for the $C(K)$ case, the constructions are based on the linear structure in a substantial way. Keeping this in mind, Theorem 2 naturally suggests the following definition:

Definition 5. Let X be a normed linear space. We say that a system $\{(x_\gamma; f_\gamma)\}_{\gamma\in\Gamma} \subset X \times X^*$ is a fundamental coordinate system if $T(x) = (f_\gamma(x))_{\gamma\in\Gamma}$ is a bounded linear operator from X to $c_0(\Gamma)$ and $x \in \overline{\operatorname{span}}\{x_\gamma;\ f_\gamma(x) \ne 0\}$ for each $x \in X$.

Note that the operator T from the definition is necessarily one-to-one and $\{f_\gamma\}_{\gamma\in\Gamma}$ is bounded (by $\|T\|$). The following corollary of Theorem 2 is now obvious.

Corollary 6. *Let X be a normed linear space with a fundamental coordinate system and such that it admits a C^k-smooth bump, $k \in \mathbb{N} \cup \{\infty\}$. Then X admits locally finite and σ-uniformly discrete $\mathcal{C}^k$-partitions of unity.*

If X has a strong Markushevich basis, then it also has a fundamental coordinate system (take normalised coordinate functionals). Thus we have an immediate generalisation of the result (i). On the other hand, as we shall see below (Corollary 9), the space JL_p, $1 < p < \infty$, has a fundamental coordinate system but it does not have a Markushevich basis ([HMVZ, Corollary 4.20]).

In connection with Corollary 6 and Corollary 4 we remark that the space $C(K)$, where K is a Ciesielski-Pol compact, does not continuously linearly inject into any $c_0(\Gamma)$ (and so it does not have a fundamental coordinate system), although it has a subspace Y isometric to $c_0(\Gamma_1)$ such that the quotient $C(K)/Y$ is isomorphic to $c_0(\Gamma_2)$, [DGZ, Theorem VI.8.8.3].

Concerning the result (i) we note that for spaces with an SPRI we do not have to rely on the full construction of a strong Markushevich basis (which is rather hard): Let $\{P_\alpha\}_{\alpha\in[\omega,\mu)}$ be an SPRI on X. For each $\alpha \in [\omega, \mu)$ put $Q_\alpha = P_{\alpha+1} - P_\alpha$, let $\{y_{\alpha n}\}_{n\in\mathbb{N}}$ be a dense subset of the separable space $Q_\alpha(X)$, and let $\{g_{\alpha k}\}_{k\in\mathbb{N}} \subset Q_\alpha(X)^*$ be separating for $Q_\alpha(X)$. Put $f_{\alpha k} = g_{\alpha k} \circ Q_\alpha/(\|g_{\alpha k} \circ Q_\alpha\| + 1)$, $\Gamma = [\omega, \mu) \times \mathbb{N} \times \mathbb{N}$, and define $T\colon X \to \ell_\infty(\Gamma)$ by $T(x)(\alpha, k, n) = \frac{1}{kn} f_{\alpha k}(x)$. Then T is clearly a bounded linear operator.

Further, set $x_\gamma = y_{\alpha n}$ for $\gamma = (\alpha, k, n) \in \Gamma$. Fix $x \in X$. Since $Q_\alpha(x) \neq 0$ if and only if there is $k \in \mathbb{N}$ such that $g_{\alpha k}(Q_\alpha(x)) \neq 0$, which is equivalent to $f_{\alpha k}(x) \neq 0$, we have

$$\begin{aligned} x \in \overline{\operatorname{span}}\{Q_\alpha(x);\ \alpha \in [\omega, \mu)\} &= \overline{\operatorname{span}}\{Q_\alpha(x);\ \alpha \in [\omega, \mu), Q_\alpha(x) \neq 0\} \\ &\subset \overline{\operatorname{span}}\big\{\overline{\{y_{\alpha n}\}_{n\in\mathbb{N}}};\ \alpha \in [\omega, \mu), \exists k \in \mathbb{N}\colon f_{\alpha k}(x) \neq 0\big\} \\ &= \overline{\operatorname{span}}\{y_{\alpha n};\ n \in \mathbb{N}, \alpha \in [\omega, \mu), \exists k \in \mathbb{N}\colon T(x)(\alpha, k, n) \neq 0\} = \overline{\operatorname{span}}\{x_\gamma;\ T(x)(\gamma) \neq 0\}. \end{aligned}$$

Note that $Q_\beta \circ Q_\alpha = 0$ for $\beta \neq \alpha$. Hence, given $\alpha \in [\omega, \mu)$ and $n \in \mathbb{N}$, we have $T(y_{\alpha n})(\beta, k, m) = \frac{1}{km} f_{\beta k}(y_{\alpha n}) = \frac{1}{km} f_{\beta k}(Q_\alpha(y_{\alpha n})) = \frac{1}{km} g_{\beta k}\big(Q_\beta\big(Q_\alpha(y_{\alpha n})\big)\big)/(\|g_{\beta k} \circ Q_\beta\| + 1) = 0$ for $\beta \neq \alpha$. Also, $|T(y_{\alpha n})(\alpha, k, m)| = \frac{1}{km}|f_{\alpha k}(y_{\alpha n})| \leq \frac{1}{km}\|y_{\alpha n}\|$. Therefore $T(y_{\alpha n}) \in c_0(\Gamma)$. Since we have seen above that $X = \overline{\operatorname{span}}\{y_{\alpha n};\ \alpha \in [\omega, \mu), n \in \mathbb{N}\}$, it follows that T maps into $c_0(\Gamma)$ and so X has a fundamental coordinate system.

We proceed by deducing the result (v) from Corollary 9 and the result (ii) from Theorem 10. We start with an easy observation.

Fact 7. *Let X be a normed linear space and $\{(x_\gamma; f_\gamma)\}_{\gamma\in\Gamma} \subset X \times X^*$. Then $x \in \overline{\operatorname{span}}\{x_\gamma;\ f_\gamma(x) \neq 0\}$ for every $x \in X$ if and only if $f \in \overline{\operatorname{span}}^{w^*}\{f_\gamma;\ f(x_\gamma) \neq 0\}$ for every $f \in X^*$.*

Proof. $\Rightarrow$ Assume that it is not true for some $f \in X^*$. Denote $A = \{\gamma \in \Gamma;\ f(x_\gamma) \neq 0\}$. By the separation theorem there is $x \in X$ such that $f(x) \neq 0$ and $f_\gamma(x) = 0$ for each $\gamma \in A$. It follows that $x \in \overline{\operatorname{span}}\{x_\gamma;\ f_\gamma(x) \neq 0\} \subset \overline{\operatorname{span}}\{x_\gamma;\ \gamma \in \Gamma \setminus A\} \subset \{f\}_\perp$, a contradiction.

$\Leftarrow$ Assume that it is not true for some $x \in X$. Denote $A = \{\gamma \in \Gamma;\ f_\gamma(x) \neq 0\}$. By the separation theorem there is $f \in X^*$ such that $f(x) \neq 0$ and $f(x_\gamma) = 0$ for each $\gamma \in A$. It follows that $f \in \overline{\operatorname{span}}^{w^*}\{f_\gamma;\ f(x_\gamma) \neq 0\} \subset \overline{\operatorname{span}}^{w^*}\{f_\gamma;\ \gamma \in \Gamma \setminus A\} \subset \{x\}^\perp$, a contradiction.

□

The first part of the next theorem is probably folklore among experts. We include the proof for the convenience of the reader.

Theorem 8. *Let X be a WCG Banach space and let $K \subset X$ be a weakly compact convex symmetric set that generates X. Then X has a strong Markushevich basis $\{(x_\gamma; f_\gamma)\}_{\gamma\in\Gamma} \subset K \times X^*$. Such a basis has the following properties: $T(f) = (f(x_\gamma))_{\gamma\in\Gamma}$ is a bounded linear operator from X^* to $c_0(\Gamma)$ and $f \in \overline{\operatorname{span}}^{w^*}\{f_\gamma;\ f(x_\gamma) \neq 0\}$ for each $f \in X^*$.*

Proof. We prove the first part by transfinite induction on $\operatorname{dens} X$. Suppose first that X is separable. Let $\{z_n\}_{n\in\mathbb{N}} \subset K$ be a dense set in K and $\{h_n\}_{n\in\mathbb{N}}$ a norming set in X^*. Note that $\overline{\operatorname{span}}\{z_n\} = X$. By [FHHMZ, Theorem 4.59] there is a Markushevich basis $\{(y_n; g_n)\}_{n\in\mathbb{N}}$ of X such that $\operatorname{span}\{y_n\} = \operatorname{span}\{z_n\}$ and $\operatorname{span}\{g_n\} = \operatorname{span}\{h_n\}$. In particular, this basis is norming. Hence by [HMVZ, Theorem 1.42] there is a strong Markushevich basis $\{(x_n; f_n)\}_{n\in\mathbb{N}}$ of X such that $\{x_n\} \subset \operatorname{span}\{y_n\} = \operatorname{span}\{z_n\}$. Since $\operatorname{span}\{z_n\} \subset \bigcup_{n\in\mathbb{N}} nK$, by scaling we may assume that $\{x_n\} \subset K$.

Now assume that $\operatorname{dens} X > \omega$ and the statement is true for all WCG spaces of density less than $\operatorname{dens} X$. By [FHHMZ, Theorem 13.6] there is a PRI $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ on X such that $P_\alpha(K) \subset K$ for each $\alpha \in [\omega, \mu]$. Denote $Q_\alpha = P_{\alpha+1} - P_\alpha$. For each $\alpha \in [\omega, \mu)$ the space $Q_\alpha(X)$ is of density at most $\operatorname{card}\alpha < \operatorname{dens} X$ and is generated by the weakly compact convex symmetric set $\frac{1}{2}Q_\alpha(K)$. Thus by the inductive hypothesis $Q_\alpha(X)$ has a strong Markushevich basis $\{(x^\alpha_\gamma; g^\alpha_\gamma)\}_{\gamma\in\Gamma_\alpha}$ such that $\{x^\alpha_\gamma\}_{\gamma\in\Gamma_\alpha} \subset \frac{1}{2}Q_\alpha(K) \subset K$. Put $f^\alpha_\gamma = g^\alpha_\gamma \circ Q_\alpha$. We claim that $\{(x^\alpha_\gamma; f^\alpha_\gamma)\}_{\alpha\in[\omega,\mu),\gamma\in\Gamma_\alpha}$ is a strong Markushevich basis of X.

Indeed, $Q_\alpha(x^\beta_\eta) = Q_\alpha(Q_\beta(x^\beta_\eta)) = 0$ and hence $f^\alpha_\gamma(x^\beta_\eta) = g^\alpha_\gamma(Q_\alpha(x^\beta_\eta)) = 0$ for $\alpha \neq \beta$. Further, $f^\alpha_\gamma(x^\alpha_\eta) = g^\alpha_\gamma(Q_\alpha(x^\alpha_\eta)) = g^\alpha_\gamma(x^\alpha_\eta) = \delta_{\gamma,\eta}$ (the Kronecker delta). Now fix any $x \in X$. Then $Q_\alpha(x) \in \overline{\operatorname{span}}\{x^\alpha_\gamma;\ \gamma \in \Gamma_\alpha\colon g^\alpha_\gamma(Q_\alpha(x)) \neq 0\} = \overline{\operatorname{span}}\{x^\alpha_\gamma;\ \gamma \in \Gamma_\alpha\colon f^\alpha_\gamma(x) \neq 0\}$. Hence $x \in \overline{\operatorname{span}}\{Q_\alpha(x);\ \alpha \in [\omega, \mu)\} \subset \overline{\operatorname{span}} \bigcup_{\alpha\in[\omega,\mu)} \overline{\operatorname{span}}\{x^\alpha_\gamma;\ \gamma \in \Gamma_\alpha\colon f^\alpha_\gamma(x) \neq 0\} \subset \overline{\operatorname{span}}\{x^\alpha_\gamma;\ \alpha \in [\omega, \mu), \gamma \in \Gamma_\alpha\colon f^\alpha_\gamma(x) \neq 0\}$. Finally, note that this strongness property implies that the biorthogonal system is total.

To prove the second part of the theorem, denote by τ the topology on X^* given by the uniform convergence on K. Put $T(f) = (f(x_\gamma))_{\gamma\in\Gamma}$ for $f \in X^*$. Then T is clearly a bounded linear operator from X^* to $\ell_\infty(\Gamma)$. Since $\|T(f)\| = \sup_{\gamma\in\Gamma}|f(x_\gamma)| \leq \sup_{x\in K}|f(x)|$, the operator T is moreover τ-$\|\cdot\|$ continuous. Further, $T(f_\alpha) \in c_{00}(\Gamma)$ for every $\alpha \in \Gamma$. By the Mackey-Arens theorem, $\overline{\operatorname{span}}^{\tau}\{f_\alpha\} = \overline{\operatorname{span}}^{w^*}\{f_\alpha\} = X^*$. Consequently, $T(X^*) = T(\overline{\operatorname{span}}^{\tau}\{f_\alpha\}) \subset \overline{\operatorname{span}}\{T(f_\alpha)\} \subset c_0(\Gamma)$. The rest follows from Fact 7.

□

We remark that the heart of the construction of a strong Markushevich basis lies in the separable case and is seriously difficult. The strongness of the PRI then arranges the rest. However, for our purpose (the second part of the previous theorem), the full strongness (and even the biorthogonality) of the Markushevich basis is not necessary. It would be sufficient to carry the required properties through the transfinite induction and use just the strongness provided by the PRI. The weak compactness is indispensable though.

Corollary 9. *Let X be a normed linear space such that X^* is WCG. Then X has a fundamental coordinate system.*

Proof. Let $\{(f_\gamma; F_\gamma)\}_{\gamma\in\Gamma} \subset X^* \times X^{**}$ be a Markushevich basis from Theorem 8. Note that $\{f_\gamma\}_{\gamma\in\Gamma}$ is bounded. Fix $\gamma \in \Gamma$. Then by the Goldstine theorem $F_\gamma \in \overline{B}^{w^*}$ for some ball $B \subset X$. Since X^* has the property C ([FHHMZ, Definition 14.32, Theorem 14.31]), by [FHHMZ, Theorem 14.37] there is a countable set $\{x_{\gamma n}\}_{n\in\mathbb{N}} \subset B$ such that $F_\gamma \in \overline{\operatorname{conv}}^{w^*}\{x_{\gamma n}\}_{n\in\mathbb{N}}$. We claim that $\{(x_{\gamma n}; \frac{1}{n} f_\gamma)\}_{\gamma\in\Gamma, n\in\mathbb{N}}$ is a fundamental coordinate system.

Indeed, $T(x) = \left(\frac{1}{n} f_\gamma(x)\right)_{\gamma\in\Gamma, n\in\mathbb{N}}$ is a bounded linear operator from X to $c_0(\Gamma\times\mathbb{N})$, since $(f_\gamma(x))_{\gamma\in\Gamma} \in c_0(\Gamma)$ by Theorem 8. Fix $x \in X$ and denote $A = \{x_{\gamma n};\ f_\gamma(x) \neq 0, n \in \mathbb{N}\}$. Theorem 8 implies that $F \in \overline{\operatorname{span}}^{w^*}\{F_\gamma;\ F(f_\gamma) \neq 0\}$ for any $F \in X^{**}$ and so

$$x \in \overline{\operatorname{span}}^{w^*}\{F_\gamma;\ f_\gamma(x) \neq 0\} \subset \overline{\operatorname{span}}^{w^*} \bigcup_{\gamma\in\Gamma:\ f_\gamma(x)\neq 0} \overline{\operatorname{conv}}^{w^*}\{x_{\gamma n}\}_{n\in\mathbb{N}} = \overline{\operatorname{span}}^{w^*} A.$$

But since $x \in X$ and $\operatorname{span} A \subset X$, this means that $x \in \overline{\operatorname{span}}^{w} A = \overline{\operatorname{span}}\, A$.

□

We note that there is a Banach space X such that it is a second dual space, it has an equivalent C^1-smooth norm, X^* is a subspace of a Hilbert-generated space (in particular a subspace of a WCG space), and there is no bounded linear one-to-one operator from X to $c_0(\Gamma)$, [AM]. Therefore there is no hope for generalising the result (v) beyond the dual being WCG using the approach above (or the original proof as well – both result in a linear injection into $c_0(\Gamma)$).

Theorem 10. *Every Banach space that belongs to a $\overline{\mathcal{P}}$-class has a fundamental coordinate system.*

Proof. Let $\mathcal{X}$ be a $\overline{\mathcal{P}}$-class and $X \in \mathcal{X}$. We use transfinite induction on $\operatorname{dens} X$. If X is separable, then we can use the existence of a strong Markushevich basis. However, this difficult result is not necessary. A direct construction is as follows: Let $\{y_n\}_{n\in\mathbb{N}} \subset X$ be dense in X and let $\{g_n\}_{n\in\mathbb{N}} \subset X^*$ be such that it separates the points of X and $\|g_n\| \leq \frac{1}{n}$. For $k, n \in \mathbb{N}$ put $x_{kn} = y_n$ and $f_{kn} = \frac{1}{n} g_k$. Then $\{(x_{kn}, f_{kn})\}_{k,n\in\mathbb{N}}$ is a fundamental coordinate system: Fix $x \in X$. Then $|f_{kn}(x)| \leq \frac{1}{nk}\|x\|$. Also, there is $m \in \mathbb{N}$ such that $g_m(x) \neq 0$ and so $x \in \overline{\operatorname{span}}\{y_n;\ n \in \mathbb{N}\} = \overline{\operatorname{span}}\{x_{mn};\ n \in \mathbb{N}\} \subset \overline{\operatorname{span}}\{x_{kn};\ f_{kn}(x) \neq 0\}$.

Now assume that $\operatorname{dens} X > \omega$ and every space in $\mathcal{X}$ of density less than $\operatorname{dens} X$ has a fundamental coordinate system. Let $\{P_\alpha\}_{\alpha\in[\omega,\mu]}$ be a PRI on X such that $P_\alpha(X) \in \mathcal{X}$ for $\alpha \in [\omega, \mu)$. Put $Q_\alpha = P_{\alpha+1} - P_\alpha$. By the inductive hypothesis, for each $\alpha \in [\omega, \mu)$ there is a fundamental coordinate system $\{(x_\gamma^\alpha; g_\gamma^\alpha)\}_{\gamma\in\Gamma_\alpha}$ on $P_\alpha(X)$ and there is $K_\alpha > 0$ such that $\{g_\gamma^\alpha\}_{\gamma\in\Gamma_\alpha} \subset B(0, K_\alpha)$. Since $Q_\alpha(X) \subset P_{\alpha+1}(X)$, we may set $f_\gamma^{\alpha+1} = \frac{1}{K_{\alpha+1}} g_\gamma^{\alpha+1} \circ Q_\alpha$ and note that $\|f_\gamma^{\alpha+1}\| \leq 2$. We claim that $\{(x_\gamma^{\alpha+1}; f_\gamma^{\alpha+1})\}_{\alpha\in[\omega,\mu), \gamma\in\Gamma_{\alpha+1}}$ is a fundamental coordinate system on X.

Indeed, the formula $T(x) = (f_\gamma^{\alpha+1}(x))_{\alpha\in[\omega,\mu), \gamma\in\Gamma_{\alpha+1}}$ clearly defines a bounded linear operator from X to $\ell_\infty(\Gamma)$, where $\Gamma = \bigcup_{\alpha\in[\omega,\mu)}\{\alpha\} \times \Gamma_{\alpha+1}$. Now fix $x \in X$ and $\varepsilon > 0$. Then the set $A = \{\alpha \in [\omega, \mu);\ \|Q_\alpha(x)\| > \varepsilon\}$ is finite. So, $|f_\gamma^{\alpha+1}(x)| \leq \frac{1}{K_{\alpha+1}}\|g_\gamma^{\alpha+1}\|\|Q_\alpha(x)\| \leq \varepsilon$ whenever $\alpha \in [\omega, \mu) \setminus A$ and $\gamma \in \Gamma_{\alpha+1}$. On the other hand, if $\alpha \in A$, then the set $\{\gamma \in \Gamma_{\alpha+1};\ |f_\gamma^{\alpha+1}(x)| > \varepsilon\} = \left\{\gamma \in \Gamma_{\alpha+1};\ |g_\gamma^{\alpha+1}(Q_\alpha(x))| > K_{\alpha+1}\varepsilon\right\}$ is finite by the definition of a fundamental coordinate system. Finally, as $Q_\alpha(x) \in P_{\alpha+1}(X)$, the assumption yields $Q_\alpha(x) \in \overline{\operatorname{span}}\{x_\gamma^{\alpha+1};\ g_\gamma^{\alpha+1}(Q_\alpha(x)) \neq 0\} = \overline{\operatorname{span}}\{x_\gamma^{\alpha+1};\ f_\gamma^{\alpha+1}(x) \neq 0\}$. Therefore $x \in \overline{\operatorname{span}}\{Q_\alpha(x);\ \alpha \in [\omega, \mu)\} \subset \overline{\operatorname{span}} \bigcup_{\alpha\in[\omega,\mu)} \overline{\operatorname{span}}\{x_\gamma^{\alpha+1};\ f_\gamma^{\alpha+1}(x) \neq 0\} \subset \overline{\operatorname{span}}\{x_\gamma^{\alpha+1};\ f_\gamma^{\alpha+1}(x) \neq 0\}$.

□

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